Friction and geometry in Lindbladian dynamics

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Mathematical Horizons for Quantum Physics 2
National University of Singapore
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joint work with Y. Avron, M. Fraas, P. Grech, O. Kenneth

Outline

Known facts, just to set the stage

Lindbladians

The linear response of Lindbladian fluxes

The linear response of Hamiltonian fluxes

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Linear response: Operators $A = A^*$, $B = B^*$

A quantum system H_0 driven by A (interaction $H_I(t) = -\phi(t)A$) responds through B

$$\langle B \rangle = \langle B \rangle_0 + f_{BA}\phi(0)$$

(f_{BA}: static response) with

$$f_{BA} = \mathrm{i} \int_0^\infty \mathrm{tr}(\mathrm{e}^{\mathrm{i}H_0 t} B \mathrm{e}^{-\mathrm{i}H_0 t} [A, \rho_0]) dt$$

(Kubo). Relates response to correlations in an equilibrium state ρ_0.

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Next: Two contrasting examples

First example: Integer quantum Hall effect (IQHE)

Driver E_ν (field), response j_μ (current)

$$\langle j_\mu \rangle = \sigma_{\mu\nu} E_\nu$$

($\sigma_{\mu\nu}$: conductivity) with Kubo formula (Thouless et al.)

$$\sigma_{\mu\nu} = \mathrm{i}\,\mathrm{tr}(P_\perp [\partial_\mu P, \partial_\nu P])$$

where P is the (1-particle density matrix of the) ground state of H_0 and $\partial_\mu P \equiv \mathrm{i}[P, x_\mu]$.

Second example: Brownian motion

Phenomenological equation:

$$\langle\vec{j}\rangle = \langle\vec{j}\rangle_{\text{drift}} + \langle\vec{j}\rangle_{\text{diffusion}} \equiv \rho\vec{v} - \gamma\vec{\nabla}\rho$$

($\rho = \rho(x)$ density, γ diffusion constant).

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($\rho = \rho(x)$ density, γ diffusion constant). Viewed as an application of linear response for homogeneous ρ: driver $\vec{F}$ (force), response $\vec{j}$ (current)

$$\langle \vec{j} \rangle_{\text{drift}} = \rho \mu \vec{F}$$

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An association of thoughts

- $$f_{BA} = \mathrm{i}\int_0^\infty \mathrm{tr}(\mathrm{e}^{\mathrm{i}H_0 t} B \mathrm{e}^{-\mathrm{i}H_0 t}[A, \rho_0])dt$$

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Outline

A remark: Fluxes (or rates), but which ones?

Some observables $\dot{X}$ (e.g. currents) are the rate of change of others, X. In Hamiltonian dynamics

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$$X \mapsto X_{\mathrm{S}}$$

retains "the part of X which pertains to S". E.g. $H \mapsto H_{\mathrm{S}} \neq H$.

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We'll see: Both answers ($(\dot{X})_S, \dot{X}_{\mathrm{S}}$) are right, but come with different meanings.

Lindblad evolution

Quantum System coupled to Bath. What is the evolution of the system?

Evolution of a mixed state $\rho = \rho_S$

$$\rho \mapsto \phi_t(\rho) = \mathrm{tr}_B\big(U_t(\rho \otimes \rho_B)U_t^*\big)$$

with joint unitary (Hamiltonian) evolution U_t ($U_{t+s} = U_t U_s$) on $\mathcal{H} \otimes \mathcal{H}_B$
Properties:

- $\mathrm{tr}\, \phi_t(\rho) = \mathrm{tr}\, \rho$
- ϕ_t completely positive
- $\phi_{t+s} = \phi_t \circ \phi_s$
  - approximately, if time scales of Bath $\ll$ time scales of System (Davies, Spohn; Jaksic, Pillet)
  - exactly, if bath is white noise (and some other cases)

Lindblad generator

- $\operatorname{tr} \phi_t(\rho) = \operatorname{tr} \rho$
- ϕ_t completely positive
- $\phi_{t+s} = \phi_t \circ \phi_s$

Generator:

$$\mathcal{L} := \frac{d\phi_t}{dt}\Big|_{t=0}$$

Theorem (Lindblad, Sudarshan-Kossakowski-Gorini 1976)
The general form of the generator is

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] + \mathcal{D}(\rho)$$

with dissipative term

$$\mathcal{D}(\rho) = \sum_\alpha 2\Gamma_\alpha \rho \Gamma_\alpha^* - \Gamma_\alpha^* \Gamma_\alpha \rho - \rho \Gamma_\alpha^* \Gamma_\alpha$$

and arbitrary bounded H and Γ_α.

Gauge transformations

The Lindbladian $\mathcal{L}$ does not determine H and Γ_α uniquely. In fact, $\mathcal{L}$ is invariant under the transformation

$$H \mapsto H + e1 - i \sum_\alpha (g_\alpha^* \Gamma_\alpha - g_\alpha \Gamma_\alpha^*), \qquad \Gamma_\alpha \mapsto \Gamma_\alpha + g_\alpha$$

($e \in \mathbb{R}, g_\alpha \in \mathbb{C}$).

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- The tangent map is

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- In examples, one choice of H as the “energy of the system” will be singled out naturally.

Families of Lindbladians

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Lindbladians $\mathcal{L}_\phi$ may depend on some control parameters $\phi = (\phi^\mu) \in M$. Can be used for

- virtual changes (next)
- to drive the system and probe its response (later)

Special case: Iso-spectral families

$$H(\phi) = U(\phi) H U^*(\phi), \qquad \Gamma_\alpha(\phi) = U(\phi)\Gamma_\alpha U^*(\phi)$$

for a unitary $U(\phi) = \mathrm{e}^{-\mathrm{i}G_\mu \phi^\mu}$ with generators $G_\mu = G_\mu^*$

The principle of virtual work in the Hamiltonian context

is a proven mean of associating observables X_μ (generalized forces) to parameter changes $\delta\phi^\mu$ by means of the induced δH:

$$X_\mu \delta\phi^\mu := \delta H$$

Examples:

- displacement $\rightarrow$ force (or strain $\rightarrow$ stress)
- angle $\rightarrow$ torque
- e.m. gauge transformation $\rightarrow$ current

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Examples:

- displacement $\rightarrow$ force (flux of momentum)
- angle $\rightarrow$ torque (flux of angular momentum)
- e.m. gauge transformation $\rightarrow$ current (flux of charge).

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$$X_\mu \delta\phi^\mu := \delta H + \mathrm{i}\sum_\alpha (\Gamma_\alpha^* \delta\Gamma_\alpha - \delta\Gamma_\alpha^* \Gamma_\alpha)$$

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$$X_\mu \delta\phi^\mu := \delta H + \mathrm{i} \sum_\alpha (\Gamma_\alpha^* \delta\Gamma_\alpha - \delta\Gamma_\alpha^* \Gamma_\alpha)$$

- is formally self-adjoint
- is gauge invariant
- for iso-spectral families ($\delta H = \mathrm{i}[H, G_\mu]\delta\phi^\mu$, $\delta\Gamma_\alpha$ ditto) the observable is a flux:

$$X_\mu = \mathcal{L}^*(G_\mu) = \mathrm{i}[H, G_\mu] + \mathcal{D}^*(G_\mu)$$

- $\mathcal{L}^*(G_\mu) =$ Hamiltonian flux $+$ dissipative flux (Bellissard)
- If the generalized force $X_\mu = 0$ determined by $U(\phi)$ vanishes, then its generator G_μ is a constant of motion (Noether).

Example: Brownian motion; charge

Lindbladian given by

$$H = \frac{p^2}{2}, \qquad \Gamma = \sqrt{\gamma} p$$

Heisenberg equation of motion

$$\dot{x} = \mathcal{L}^*(x) = p, \qquad \dot{p} = \mathcal{L}^*(p) = 0$$

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Iso-spectral family $H(\phi)$ obtained by e.m. gauge trsf $G = \chi(x)$:

$$X = \mathcal{L}^*(\chi) = \frac{1}{2}\{p, \nabla\chi\} + \gamma\Delta\chi$$

read in terms of

$$\chi = \int \rho(x)\chi(x)dx, \qquad X = \int j(x) \cdot \nabla\chi(x)\, dx$$

$$\rho(x_0) = \delta(x - x_0), \qquad j(x_0) = \frac{1}{2}\{p, \rho(x_0)\} - \gamma(\nabla\rho)(x_0)$$

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Note:

- $j = j_{\mathrm{drift}} + j_{\mathrm{diffusion}}$ as operators
- $X = \dot{\chi}$ is just $\dot{\rho} = -\nabla \cdot j$: local charge conservation

Example: Fermions on a lattice; charge

Lindbladian given by

$$H = \sum_{j\in\mathbb{Z}} (a_{j+1}^* a_j + a_j^* a_{j+1} - \mu a_j^* a_j), \qquad \Gamma_j^- = \sqrt{\gamma_-} a_j, \qquad \Gamma_j^+ = \sqrt{\gamma_+} a_j^*$$

$\Gamma_j^\pm$ fills holes/removes particle at site j

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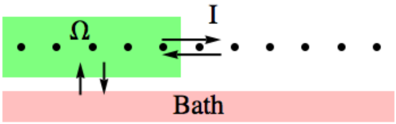
Charge in the left half-lattice: $Q = \sum_{j\le 0} a_j^* a_j$



Current

$$\dot{Q} = \mathcal{L}^*(Q) = \mathrm{i}[H, Q] + \mathcal{D}^*(Q)$$

$$\mathrm{i}[H, Q] = \mathrm{i}(a_1^* a_0 - a_0^* a_1), \quad \mathcal{D}^*(Q) = 2\sum_{j\le 0} (\gamma_+ a_j a_j^* - \gamma_- a_j^* a_j)$$

Only Hamiltonian current is local; dissipative is not.

Example: damped harmonic oscillator; momentum

Lindbladian given by

$$H = a^* a, \qquad \Gamma_- = \sqrt{\gamma}_- a, \qquad \Gamma_+ = \sqrt{\gamma}_+ a^*, \qquad (\gamma_- > \gamma_+ > 0)$$

$\Gamma_\pm$: exciting and damping.

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$$H = a^*a, \qquad \Gamma_- = \sqrt{\gamma_-}\,a, \qquad \Gamma_+ = \sqrt{\gamma_+}\,a^*, \qquad (\gamma_- > \gamma_+ > 0)$$

$\Gamma_\pm$: exciting and damping.
Iso-spectral family $H(\phi)$ obtained by translations $G = p$

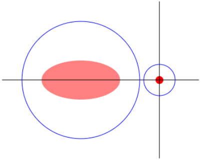
$$\dot{p} = \mathcal{L}^*(p) = \mathrm{i}[H, p] + \mathcal{D}^*(p) \equiv -x - (\gamma_- - \gamma_+)p$$

spring force and friction.

The spectrum of Lindbladians

Lindbladians $\mathcal{L}$ generate contractions. Thus

- The spectrum $\sigma(\mathcal{L})$ is contained in the complex left half-plane
- $\ker \mathcal{L} \cap \operatorname{ran} \mathcal{L} = \{0\}$



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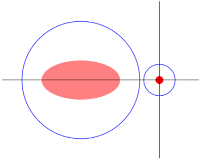

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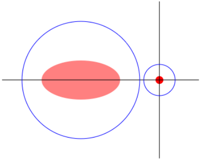

Assumption (gap condition). 0 is a discrete point in the spectrum.

$\mathcal{P}$, $\mathcal{Q}$: complementary (super) projections associated to $\ker \mathcal{L} \oplus \operatorname{ran} \mathcal{L}$

$$\mathcal{P} = (2\pi\mathrm{i})^{-1} \oint (z - \mathcal{L})^{-1} dz$$

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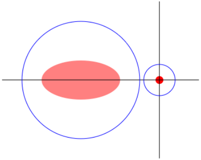
Lindbladians $\mathcal{L}$ generate contractions. Thus

- The spectrum $\sigma(\mathcal{L})$ is contained in the complex left half-plane
- $\ker \mathcal{L} \cap \operatorname{ran} \mathcal{L} = \{0\}$



Assumption (gap condition). 0 is a discrete point in the spectrum.

$\mathcal{P}$, $\mathcal{Q}$: complementary (super) projections associated to $\ker \mathcal{L} \oplus \operatorname{ran} \mathcal{L}$

$$\mathcal{P} = (2\pi\mathrm{i})^{-1} \oint (z - \mathcal{L})^{-1} dz$$

Stationary states: $\sigma = \mathcal{P}\sigma$. In fact $\dot{\sigma} = \mathcal{L}\sigma = 0$.

Lindbladians with unique stationary states

Let σ be the stationary state ($\sigma \geq 0$, $\operatorname{tr} \sigma = 1$). Then

$$\mathcal{P}(\rho) = \sigma \operatorname{tr} \rho, \qquad \mathcal{P}^*(X) = \mathbb{1} \cdot \operatorname{tr}(X\sigma)$$

- Indeed: $\mathcal{P}(\sigma) = \sigma$, $\mathcal{P}^*(\mathbb{1}) = \mathbb{1}$
- $\mathcal{P}$ depends on σ, not on $\mathcal{L}$
- Example: damped harmonic oscillator: σ is thermal with $\beta = \log(\gamma_-/\gamma_+)$

Dephasing Lindbladians

Recall the dissipative term:

$$\mathcal{D}(\rho) = \sum_\alpha 2\Gamma_\alpha \rho \Gamma_\alpha^* - \Gamma_\alpha^* \Gamma_\alpha \rho - \rho \Gamma_\alpha^* \Gamma_\alpha$$

The Lindbladian is dephasing if $\Gamma_\alpha = \Gamma_\alpha(H)$. Then

$$\mathcal{L}(P) = -\mathrm{i}[H, P] + \mathcal{D}(P) = 0$$

for any spectral projection P of H.

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for any spectral projection P of H. Let $H = \sum_j e_j P_j$ (spectral decomposition). Then

$$\mathcal{P}(\rho) = \sum_{j} P_j \rho P_j, \qquad \mathcal{P}^{*}(X) = \sum_{j} P_j X P_j$$

- Stationary states $\sigma = \mathcal{P}(\sigma)$ are the states obtained after a projective measurement ("non-demolition") of H
- If $\dim P_j = 1$, the stationary states form a simplex.

Adiabatic response

Adiabatically changing controls $\phi = \phi(s)$ where $s = \varepsilon t$ is the slow time. Evolution equation for the state ρ is

$$\varepsilon \frac{d\rho}{ds} = \mathcal{L}_\phi \rho.$$

with initial state that is an instantaneous equilibrium state $\sigma(0)$.

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Theorem Under the gap assumption the solution with initial condition the stationary state $\sigma(0)$ is

$$(\mathcal{P}\rho)(s) = \sigma(s) + \begin{cases} 0 & \text{if } \dim \mathcal{P} = 1 \\ O(\varepsilon) & \text{if } \dim \mathcal{P} \geq 2; \end{cases}$$

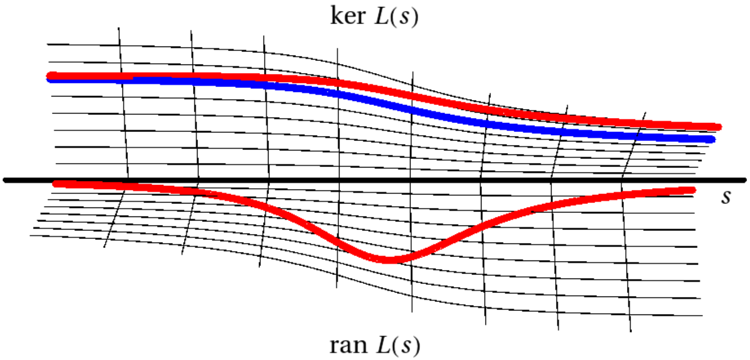
$$(\mathcal{Q}\rho)(s) = \varepsilon \mathcal{L}^{-1} \dot{\sigma}(s) + O(\varepsilon^2),$$

where $\sigma(s)$ is the corresponding integral of parallel transport $\mathcal{P}\dot{\sigma} = 0$, i.e. $\dot{\sigma} = \mathcal{Q}\dot{\sigma}$.

$\mathcal{L}^{-1}(\dot{\sigma})$ is well defined since $\dot{\sigma} \in \operatorname{ran} \mathcal{L}$ by parallel transport.

The theorem as a picture

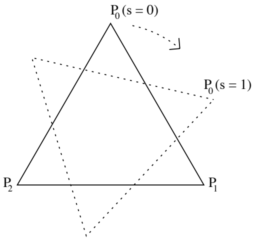
Distinct evolutions on $\ker\mathcal{L}$ and $\operatorname{ran}\mathcal{L}$:



Blue: Parallel transport. Red: including $O(\varepsilon)$ corrections

The picture in the Dephasing Lindbladian case

The instantaneous stationary states form a simplex (triangle). Extreme points represent the spectral projections $P_i(s)$, $i = 1, 2, 3$.



The motion in $\ker \mathcal{L}$:

- To order ε^0: Parallel transport rotates the triangle as a rigid body
- To order ε^1: Irreversible motion away from the vertex

Outline

The linear response of Lindbladian fluxes

Expectation of flux $\mathcal{L}^*(G_\mu)$

$$\mathrm{tr}\big(\mathcal{L}^*(G_\mu)\rho\big)(s) = f_{\mu\nu}(\phi)\,\varepsilon\dot{\phi}^\nu + O(\varepsilon^2)$$

with response matrix $f_{\mu\nu}$.

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Remarks:

- Hypothesis for free if Lindbladian dephasing, or with unique stationary state.
- Formula expresses geometric magnetism (Berry& Robbins)
- If σ is a projection P, then

$$f_{\mu\nu} = -i\,\mathrm{tr}(P[\partial_\mu P, \partial_\nu P])$$

(adiabatic curvature)

Proof

$$\begin{aligned}\mathrm{tr}(\mathcal{L}^*(G_\mu)\rho) &= \mathrm{tr}(\mathcal{Q}^*\mathcal{L}^*(G_\mu)\rho) = \mathrm{tr}(\mathcal{L}^*(G_\mu)\mathcal{Q}\rho) \\ &\cong \varepsilon\,\mathrm{tr}(\mathcal{L}^*(G_\mu)\mathcal{L}^{-1}\dot\sigma) = \varepsilon\,\mathrm{tr}(G_\mu\dot\sigma)\end{aligned}$$

Use

$$\dot\sigma = (\partial_\nu\sigma)\dot\phi^\nu, \qquad \partial_\nu\sigma = -i[G_\nu,\sigma]$$

Result:

$$f_{\mu\nu} = -i\,\mathrm{tr}([G_\mu,G_\nu]\sigma)$$

Towards an example: Landau Hamiltonian

Model of a (single) quantum particle in the plane under the influence of a uniform magnetic field.

$$H = D^*D, \qquad (D = -\mathrm{i}\partial_1 + \partial_2 + Bx_2 \equiv v_1 + \mathrm{i}v_2)$$

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- Particle in the ground state. Driver $\vec{E}$ (electric field), response $\vec{v}$ (velocity)

  $$\langle v_\mu \rangle = f_{\mu\nu} E_\nu$$

  with Hall mobility

  $$f_{\mu\nu} = B^{-1} \varepsilon_{\mu\nu}$$

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- Gas of independent particles in the ground state at density ρ. Conductivity $\sigma_{\mu\nu} = \rho f_{\mu\nu}$
- Gas filling the ground state: $\rho = B/2\pi$.

  $$\sigma_{\mu\nu} = (2\pi)^{-1} \varepsilon_{\mu\nu}$$

  Hall conductivity is quantized.

An example: IQHE and friction

Landau Lindbladian in the plane (x_1, x_2) given by

$$H = D^*D, \qquad (D = -\mathrm{i}\partial_1 + \partial_2 + Bx_2 \equiv v_1 + \mathrm{i}v_2)$$

$$\Gamma_- = \sqrt{\gamma_-}D, \qquad \Gamma_+ = \sqrt{\gamma_+}D^*, \qquad (\gamma_- > \gamma_+ \geq 0)$$

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- $[v_1, v_2] = \mathrm{i}B$
- family U_ϕ ($\phi = (\phi_1, \phi_2)$) generated by G_μ. Has

$$\mathcal{L}^*(G_\mu) = -\mathcal{L}^*(x_\mu), \qquad U_\phi v_\mu U_\phi^* = v_\mu - \phi_\mu$$

Response is (minus) velocity; driving $-\dot{\phi}$ is electric field.
Hence $f_{\mu\nu}$ is mobility.

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Theorem states:

$$f_{\mu\nu} = -i\,\mathrm{tr}([G_\mu, G_\nu]\sigma) = B^{-1}\varepsilon_{\mu\nu}$$

Hall mobility is quantized!

Outline

Known facts, just to set the stage

Lindbladians

The linear response of Lindbladian fluxes

The linear response of Hamiltonian fluxes

Groundstate geometry

Family of Hamiltonians $H(\phi)$ with control parameters $\phi = (\phi^1, \ldots, \phi^n) \in M$. E.g. M is plane, sphere, torus,...

Gap above ground state (possibly degenerate)

Geometric data associated to ground state projection $P(\phi)$

- curvature 2-form ω

$$\omega_{\mu\nu} = -\mathrm{i}\ \mathrm{tr}(P[\partial_\mu P, \partial_\nu P])$$

  (satisfies $d\omega = 0$, hence a symplectic form if non-degenerate)

- Fubini-Study metric g

$$g_{\mu\nu} = \mathrm{tr}(\partial_\mu P)(\partial_\nu P)$$

with $\partial_\mu = \partial \cdot / \partial \phi^\mu$

Response coefficients $f_{\mu\nu}$

Observables: Hamiltonian fluxes $F_\mu = \partial_\mu H = \mathrm{i}[H, G_\mu]$, conjugate to ϕ^μ.

States

- $\langle\cdot\rangle_0$: Expectation in the ground state $P(\phi)$
- For time-dependent controls $\phi(t)$
  $\langle\cdot\rangle$: Expectation in the state evolved by means of $H(\phi(t))$.

For slowly varying controls

$$\delta\langle F_\mu\rangle \equiv \langle F_\mu\rangle - \langle F_\mu\rangle_0 = f_{\mu\nu}\dot{\phi}^\nu$$

Result:

$$f_{\mu\nu} = \omega_{\mu\nu}$$

Linear response: The dephasing Lindbladian case

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  Then
  $$\mathcal{L}|j\rangle\langle i| = \lambda_{ji}|j\rangle\langle i|$$
  - $-\operatorname{Im}\lambda_{j0}$: excitation energy of $|j\rangle$ (over $|0\rangle$)
  - $-\operatorname{Re}\lambda_{j0}$: dephasing rate (loss of phase coherence) between $|j\rangle$ and $|0\rangle$.

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  - $-\operatorname{Re}\lambda_{j0}$: dephasing rate (loss of phase coherence) between $|j\rangle$ and $|0\rangle$.
- There is proportionality:
  $$\operatorname{Re}\lambda_{j0} = \gamma \operatorname{Im}\lambda_{j0}, \qquad (\gamma > 0)$$

Example of such a Dephasing Lindbladian

- System Hamiltonian A on $\mathcal{H}_S$
- Bath: $x \in \mathbb{R}$ pointer position. Hilbert space: $\mathcal{H}_B = L^2(\mathbb{R}_x)$

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$$H = A \otimes B \qquad \text{on } \mathcal{H}_S \otimes \mathcal{H}_B$$

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Then $\phi_{t+s} = \phi_t \circ \phi_s$ and $\mathcal{L}$ is of dephasing form w.r.t. A. Moreover, the rate at which different eigenstates of A loose phase coherence is proportional to their energy difference.

Generalized conductances

$$\delta\langle F_\mu\rangle \equiv \langle F_\mu\rangle - \langle F_\mu\rangle_0 = f_{\mu\nu}\dot{\phi}^\nu$$

$$\frac{d}{dt}\langle H\rangle = f_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu$$

Result:

$$f = (1+\gamma^2)^{-1}(\gamma g + \omega)$$

Decomposition into dissipative (symmetric) and reactive (antisymmetric) parts

$$f_{\mu\nu} = f_{(\mu,\nu)} + f_{[\mu,\nu]}$$

Hence

$$f_{(\mu,\nu)} = \frac{\gamma}{1+\gamma^2} g_{\mu\nu} \qquad f_{[\mu,\nu]} = \frac{1}{1+\gamma^2}\omega_{\mu\nu}$$

both affected by dephasing γ.

Kähler structure

A manifold M with metric g and symplectic form ω is almost Kähler if $J := g^{-1}\omega$ (mapping vectors to vectors) is an almost complex structure:

$$J^2 = -1$$

Equivalently,

$$\omega^{-1} g = -g^{-1}\omega \qquad (*)$$

M is Kähler if, in addition, M is a complex manifold w.r.t. J.

Examples: 1) $\mathbb{C}P^{N-1}$ (the rays of an N-dimensional Hilbert space) is Kähler.
2) Manifold $M \ni \phi$ of controls, P of rank 1: g, ω are pull-backs by way of $P : M \to \mathbb{C}P^{N-1}$. Iff (*) holds, then $P(M) \subset \mathbb{C}P^{N-1}$ is a complex submanifold. Hence $P(M)$ and M are Kähler

Criterion: Let $P = |\psi\rangle\langle\psi| / \langle\psi|\psi\rangle$. If

$$\partial_{\bar{z}_j} |\psi\rangle = 0 \qquad \text{(Cauchy-Riemann)}$$

w.r.t. complex coordinates z_j, then M is Kähler

Generalized resistances

$$\dot{\phi}^{\nu} = (f^{-1})^{\mu\nu} \delta \langle F_{\nu} \rangle$$

If M is Kähler, then

$$f^{-1} = \gamma g^{-1} + \omega^{-1}$$

and the reactive resistance is immune to dephasing γ.

Indeed

$$f = (\gamma^2 + 1)^{-1} (\gamma g + \omega)$$

and

$$(\gamma g^{-1} + \omega^{-1})(\gamma g + \omega) = \gamma^2 + 1 + \gamma (g^{-1}\omega + \omega^{-1} g) = \gamma^2 + 1$$

Examples

The Hamiltonians are obtained by unitary families.

1) Harmonic oscillator

$$H(\zeta,\mu) = \frac{\omega}{2}((p-\mu)^2 + (x-\zeta)^2 - 1)$$

with ground states $P(\zeta,\mu)$ (coherent states):
$M = \mathbb{C} \ni \zeta + \mathrm{i}\mu =: z$

$$\psi(z;x) = \mathrm{e}^{\mu^2}\mathrm{e}^{\mathrm{i}\mu x}\mathrm{e}^{-(x-\zeta)^2/2} = \mathrm{e}^{-(x-z)^2/2}$$

Analytic in z.

Examples (cont.)

2) Spin $1/2$

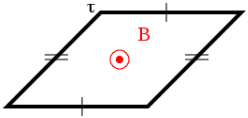
$$H(\hat{e}) = \hat{e} \cdot \vec{\sigma} \quad (\hat{e} \in S^2)$$

with ground state $P(\hat{e})$ (spin down $|-\hat{e}\rangle$): $M = S^2 \ni \hat{e}$
(Riemann sphere, z stereographic coordinate)

$$\psi(z) = \begin{pmatrix} -1 \\ z \end{pmatrix}$$

Analytic in z.

Examples (cont.)



3) Let $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C}$ define the torus $\mathbb{T} = \mathbb{R}^2/(\mathbb{Z} + \tau\mathbb{Z})$, $\vec{r} = x1 + y\tau$. Landau Hamiltonian $H(\phi_1, \phi_2)$ on $\mathbb{T}$ with boundary conditions ϕ_1, ϕ_2 and flux 2π. Then $M = \mathbb{R}^2 \ni (\phi_1, \phi_2)$ with complex structure τ (coordinate $\phi = \phi_1 - \phi_2/\tau$)

$$\psi(\phi; \vec{r}) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}\pi n x} \mathrm{e}^{\mathrm{i}\pi(y+n+\phi)^2}$$

Analytic in ϕ.
Reactive resistance is Hall resistance.

Summary

- Lindbladians describe the dynamics of open systems
- Interesting observables: Fluxes, related to virtual work
- Linbladian fluxes $\neq$ Hamiltonian fluxes
- Adiabatic theory for Lindbladians, and linear response.
- Linear response theory can be geometric (adiabatic curvature/Fubini-Study metric)
- Linear response coefficients can be quantized in presence of dissipation

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Or for short: There is geometry in dissipation (decoherence, dephasing ..)