On the role of entanglement in work extraction

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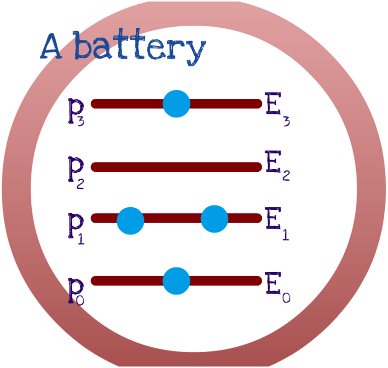
Table of contents

Introduction
- One battery
- N batteries

Work and Entanglement
- Setting the problem
- Indirect paths
- Direct paths
- Direct Paths: applications

Conclusions

A battery



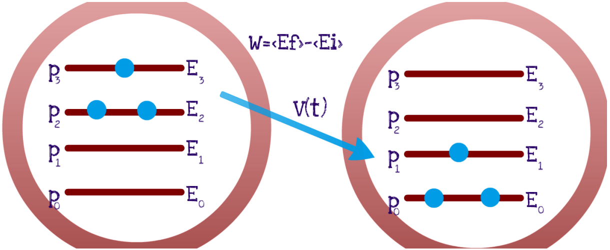
$$H_{\text{int}} = \sum_i E_i |i\rangle\langle i|$$

$$\rho = \sum_i p_i |i\rangle\langle i|$$

Work Extraction

- Physical Picture $\rightarrow$ a controlled external field $V(t)$ acts on ρ during $t \in (0, \tau)$.
- Hamiltonian $\rightarrow H = H_{\text{int}} + V(t)$ and corresponding $U(t)$
- Definition of *work* [1]:

$$W = \mathrm{Tr}[\rho H_{\text{int}}] - \mathrm{Tr}[U(\tau)\rho U^{\dagger}(\tau) H_{\text{int}}]$$



- the entropy of ρ is preserved

[1] W. Pusz and S.L. Woronowicz, CMP 58, 273 (1978).

Passive and Thermal States

Passive state

- Consider U' minimizing $\mathrm{Tr}[U\rho U^{\dagger} H_{\mathrm{int}}]\ \forall U$ (i.e., maximal work) $\rightarrow \sigma_{\mathrm{p}} \equiv U'\rho U'^{\dagger}$ is a passive state
- σ_{p} has the form: $\sigma_{\mathrm{p}} = \sum_i p_i |i\rangle\langle i|$ with $p_{i+1} \leq p_i$ where $\{p_i\}$ are the eigenvalues of ρ

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Thermal State

- The state minimizing $\mathrm{Tr}[\Pi H_{\mathrm{int}}]$ $\forall \Pi$ s.t. $S(\Pi) = S(\rho)$ is the thermal state:

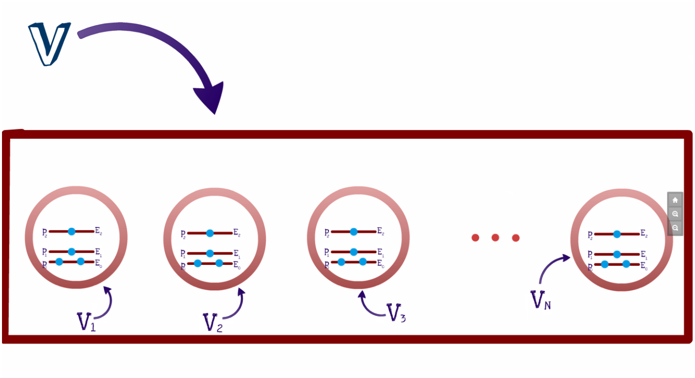
$$\sigma^{\mathrm{th}} = e^{-\beta H_{\mathrm{int}}}$$

- In general it holds

$$\mathrm{Tr}[\rho H_{\mathrm{int}}] \geq \mathrm{Tr}[\sigma_{\mathrm{p}} H_{\mathrm{int}}] \geq \mathrm{Tr}[\sigma^{\mathrm{th}} H_{\mathrm{int}}]$$

Global operations and work

Consider a set of N batteries: $H = \sum_i H_i$:



Fact: *global unitary operations are capable of extracting more work than local operations.*[2]

[2] W. Pusz and S.L. Woronowicz, CMP 58, 273 (1978)
R. Alicki and M. Fannes, PRE 87, 042123 (2013).

Global operations and work II

Examples:

- set of identical passive-but-not-thermal states $\Pi = \otimes^N \sigma_p$:
- set of thermal states: $\otimes^N e^{-\beta H_i} = e^{-\beta \sum_i H_i} \rightarrow$ no extractable work!
- set of correlated thermal states (ex. microcanonical state)

Motivation

Motivation

- Initial and final state of the set of N batteries (after maximal work extraction):

$$\Omega_i = \mathrm{diag}\{p_1, ..., p_{d^N}\} \quad \stackrel{U_N}{\rightarrow} \quad \Omega_f = \mathrm{diag}\{p_{\sigma(1)}, ..., p_{\sigma(d^N)}\},$$

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Questions

- Does the state get entangled during the process?
- Is there a way to avoid the generation of entanglement?
- What is the relation between entanglement production and work extraction?

Approach

- We aim to describe the reordering of the elements of Ω as a continuous time evolution.
- Any reordering can be made as a set of transpositions.
- Consider the transposition exchanging the populations of $|\alpha\rangle$ and $|\beta\rangle$. We choose a controlled potential $V(t)$ generating a unitary evolution $U(t)$ such that

$$U(0) = \mathcal{I}$$
$$U(\tau) = \sum_{\gamma \neq \{\alpha,\beta\}} |\gamma\rangle\langle\gamma| + |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|$$

- We measure the entanglement of $U(t)\Omega U^{\dagger}(t)$ for any t.

Bypassing entanglement

- Consider the exchange of populations of states $|0...0\rangle$, $|1...1\rangle$.
- Consider the indirect path:

$$|0...0\rangle \overset{U_1(t)}{\leftrightarrow} |10...0\rangle \overset{U_2(t)}{\leftrightarrow} |110...0\rangle \overset{U_3(t)}{\leftrightarrow} \ldots \overset{U_N(t)}{\leftrightarrow} |1...1\rangle$$

- Notice $U_i(t)$'s are global operations
- However $U_i(t)$'s do not entangle basis product states!
- But $2N - 1$ steps are required for just one transposition

Direct paths

- *direct path* $\rightarrow$ $V(t)$ only acts on the states from which we extract work.
- We measure multipartite entanglement (k-separability of $\Omega(t)$) through the entropy vector formalism.[3]
- For example, consider a tripartite state ABC:

$$\begin{pmatrix} S(A|BC) \\ S(B|AC) \\ S(C|AB) \end{pmatrix} \rightarrow \begin{cases} \text{3 zeros} \rightarrow \text{fully separable} \\ \text{1 zeros} \rightarrow \text{bipartite entanglement} \\ \text{no zeros} \rightarrow \text{tripartite entanglement} \end{cases}$$

[3] M. Huber and J. I. de Vicente, PRL. 110, 030501 (2013)
M. Huber, M. Perarnau-Llobet, J. I. de Vicente arXiv:1303.4686 (2013).

Applications I: An infinite set of identical batteries

In the limit $N \to \infty$, one can asymptotically reach[4]:

$$W/N = \mathrm{Tr}\left[H_{\text{int}}(\sigma_p - \sigma^{\text{th}})\right]$$

Following a direct path the state is at most l-separable if

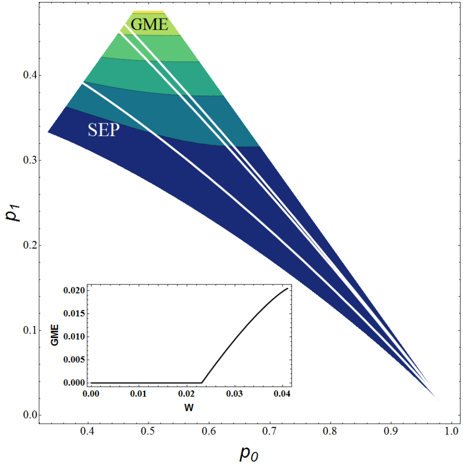
$$S(\sigma^{\text{th}}||\sigma_p) \geq \frac{\ln\left[1 + 2\gamma + 2\sqrt{\gamma + \gamma^2}\right]}{N}, \qquad \gamma = 2^{N-1} - 2^{l-1} + 1$$

Concretely,

- entanglement is present if $S(\sigma^{\text{th}}||\sigma_p) \geq \ln[3 + 2\sqrt{2}]/N$
- N-partite entanglement if $S(\sigma^{\text{th}}||\sigma_p) \geq \ln[2]$

[4] R. Alicki and M. Fannes, PRE 87, 042123 (2013)

Applications II: Four 3-level passive batteries



more work, more entanglement!

Applications III: The microcanonical state

Exchange of the population of states with energy E_0 (and population $1/N$)

$$|\overbrace{1...1}^{k}\overbrace{0...0}^{N-k}\rangle$$

to states with the minimal possible energy (with population 0), e.g.,

$$|0...0\rangle$$

Any direct path will generate k-partite entanglement.

Summing up

- Direct paths: entanglement is widely present, more work requires more (multipartite) entanglement.
- Indirect pahts: maximal work can be extracted without generating any entanglement, but more time is required.
- Our analysis thus suggests that: *entanglement generation is related to power of work extraction.*
- More info: arXiv:1303.4686