Localization in the Ground State of Mean Field Models with Random Potentials

Michael Bishop

University of California, Davis, Department of Mathematics

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National University of Singapore

Random Schrödinger operator

On the set of states $\ell^2(\{0, 1, \ldots, L, L+1\})$, the **discrete Schrödinger operator** is defined as:

$$H = -\Delta + V$$

where $-\Delta$ is the discrete Laplacian defined by

$$(-\Delta\phi)(x) = 2\phi(x) - \phi(x-1) - \phi(x+1)$$

and V is a multiplication operator $(V\phi)(x) = V(x)\phi(x)$. The operator becomes random when the set of potentials V_ω is given a probability measure.

The **Ground State Energy**, E_0 is the minimizer of the associated energy functional

$$E_0 = \min_{\{\|\phi\|=1\}} \sum_x |\phi'(x)|^2 + V(x)|\phi(x)|^2$$

Typical States in Bernoulli Disorder

In this case, an independent, identical Bernoulli distribution at each $x \in \{0, 1, \ldots, L, L+1\}$.

$$P[V(x) = 0] = p$$
$$P[V(x) = b] = q = 1 - p$$

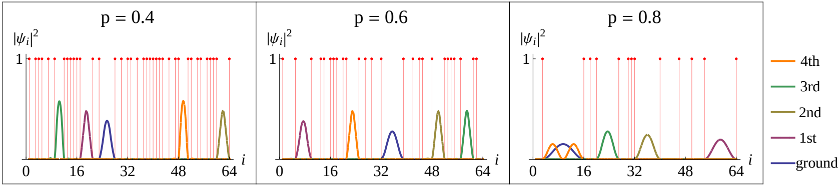

Figure: Densities of the lowest eigenstates of one-dimensional system with Bernoulli potential. These states localize to large intervals of zero potential. - J. Stasińska, P. Massignan, Institut de Ciéncies Fotóniques (ICFO)

Cold Atom Experiments

Bose Einstein condensates were experimentally realized by Cornell and Wieman at Colorado in 1995.

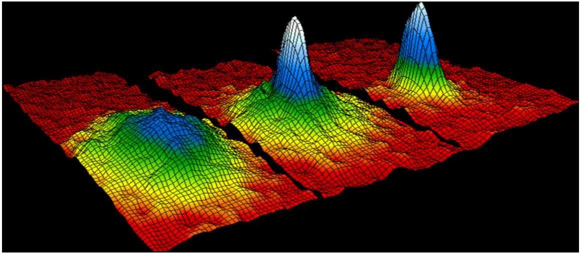

Figure: The velocity profile of rubidium atoms.[1]

[1] http://www.bec.nist.gov/gallery.html

Interacting Systems

A system of $N+1$ particles with bosonic symmetry is described by a state in

$$\otimes_{symm}^{N+1}\ell^2(\{0,\ldots,L+1\})$$

with Dirichlet boundary conditions. The Schrödinger operator with interactions is

$$\sum_i -\Delta_i + V_i + \sum_{j\leq i} U(x_i,x_j)$$

with $-\Delta_i + V_i$ acting on the i-th coordinate and $U(x_i,x_j)$ describing the interaction between the i-th and j-th particle, which will be 'soft core' interactions of the form

$$U(x_i,x_j) = g\delta_{x_i,x_j}$$

Random Potentials and Interactions

If $g > 0$, the interactions are repulsive.

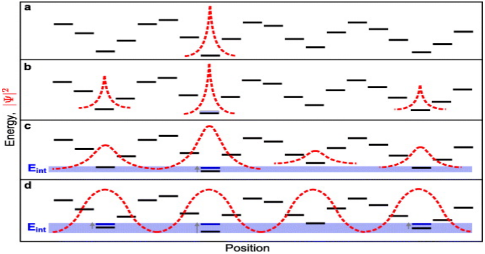

Figure: Effects of Interaction. [2]

A possible method for distinguishing cases is through the size of the spatial support of the state.

[2] B. Deissler, E. Lucioni, M. Modugno, G. Roati, L. Tanzi, M. Zaccanti, M. Inguscio, G. Modugno. **Correlation function of weakly interacting bosons in a disordered lattice**. *New* Journal of Physics, 13:023020, 2011.

Mean-field Approximation

Because the tensor space is large, it is common to approximate a state by Gross-Pitaevskii mean-field where each particle is in the same single particle state.

$$\Phi(x_0, \ldots, x_N) = \prod_i \phi(x_i) \tag{1}$$

The operator restricted to mean-field states becomes the nonlinear Schrödinger operator

$$(N+1)(-\Delta_i + V_i) + \frac{gN(N+1)}{2}|\phi(x)|^2 \tag{2}$$

Mean-field Ground State in Bernoulli Potential

The mean-field ground state energy in Bernoulli potential is the energy (per particle) minimizer

$$E_0 = \min_{\{\|\phi\|=1\}} \sum_x |\phi'(x)|^2 + V(x)|\phi(x)|^2 + \frac{gN}{2}|\phi(x)|^4 \tag{3}$$

with the ground state ϕ_0 being the corresponding function.

How does the interaction term change the spatial support of the ground state?

Interaction Energy

Given a multiparticle state where each particle is in the single particle state ϕ with energy per particle $E' = E(\phi)$, for $\epsilon \in (0, 1)$, define the set

$$X_{>\epsilon}(\phi) = \{x : |\phi(x)| > \frac{\epsilon}{L^{1/2}}\} \tag{4}$$

The interaction energy is bounded below using

$$\sum_x |\phi|^4 \geq \frac{\left(\| \phi|_{X_{>\epsilon}(\phi)} \|^2\right)^2}{\#(X_{>\epsilon}(\phi))} \tag{5}$$

This lower bound can be achieved by a constant function. For the interaction energy to be less than E', it is necessary that:

$$\frac{gN\| \phi|_{X_{>\epsilon}} \|^4}{2\#(X_{>\epsilon}(\phi))} \leq E' \tag{6}$$

Interaction Energy

Since the interaction energy to be less than E',

$$\frac{gN\|\phi|_{X_{>\epsilon}}\|^2}{2\mu(X_{>\epsilon}(\phi))} \leq E' \tag{7}$$

Rearranged:

$$\mu(X_{>\epsilon}(\phi)) \geq \frac{gN(1-\epsilon^2)^2}{2E'} \tag{8}$$

Localization

For localization to hold in large system limits, gN/E' needs to be the same order as the localization length.

Physicists are interested in cases where the particle density is approximately constant, $N \approx \rho L^d$, and g is a controlled variable. In this case, $\mu(X_{>\epsilon}(\phi)) \approx O(\frac{g\rho L^d}{E'})$.

Ground State Bernoulli Case

What does the ground state look like in a Bernoulli potential on a one dimensional lattice in the large system limit for small g?

The energy functional is

$$E(\phi) = \sum_{x} |\phi'(x)|^2 + V(x)|\phi(x)|^2 + \frac{g\rho L}{2}|\phi(x)|^4 \tag{9}$$

where $P[V = 0] = p$ and $P[V = b] = q = 1 - p$, L is the lattice length, and ϕ' is the discrete derivative. The finite volume ground state ϕ_0 minimizes this functional and E_0 is the associated energy.

Ground State Bernoulli Case

What is the nature of the ground state for small $g\rho$?

Three factors:

1) Potential barriers should be sharp.

2) Kinetic energy favors long intervals of zero potential.

3) Interaction energy favors occupation of many sites.

Ground State Bernoulli Case

The ground state is approximated by sine waves on intervals of zero longer than $\tilde{L} = \log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)$. Its energy is bounded above by

$$\frac{\pi^2}{\left(\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)\right)^2} + \frac{3g\rho L}{4\mu(X_{>0})}$$

Ground State Bernoulli Case

The intervals of zero potential are distributed geometrically:

$$P[L_i = x] = qp^{x-1}$$

The number of sites on intervals longer than $\tilde{L}$ is approximately

$$(pqL) * g\rho \log_p(g\rho) * \left(\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)\right)$$

The upper bound on the ground state energy is

$$E_0 \leq \frac{\pi^2}{\left(\log_p(g\rho)\right)^2} + \frac{3}{4pq\left(\log_p(g\rho)\right)^2}$$

Ground State Bernoulli Case

Applying the inequality

$$\mu(X_{>\epsilon}(\phi_0)) \geq \frac{gN(1-\epsilon^2)^2}{2E'} \tag{10}$$

Using this upper bound, the ground state must occupy

$$\mu(X_{>\epsilon}(\phi_0)) \geq O\left(g\rho L\left(\log_p(g\rho)^2\right)\right) \tag{11}$$

Ground State Bernoulli Case

The steps for showing lower bound for the ground state energy [3]:

- The ground state ϕ_0 exists but is unknown. It has the same complex phase at each site:

$$E(\phi) = \sum_x |\phi'(x)|^2 + V(x)|\phi(x)|^2 + \frac{g\rho L}{2}|\phi(x)|^4 \tag{12}$$

- The space is split up into four sets: M_b, sites of high potential; M_{long}, sites on intervals longer than $\log_p(g\rho)$; as well as M_{light} and M_{heavy}.

[3] MB, J. Wehr. Ground state energy of the one-dimensional discrete random Schrödinger operator in Bernoulli potential. *Journal of Statistical Physics*, 147:529-541, 2012

Ground State Bernoulli Case

- M_{light} is the set of intervals where ϕ_0 is small relative to its boundary conditions. M_{heavy} is the set of intervals where ϕ_0 is large relative to its boundary conditions, $m_i\delta^L$ and $m_i\delta^R$. They are determined by a parameter $\gamma \in (0, 1)$.

  An interval is **heavy** if

$$\max(\delta^L, \delta^R) \leq \frac{\gamma}{\sqrt{L_i}}$$

- The norm of ϕ_0 restricted to M_{heavy} goes to one as $g\rho \to 0$.
- The kinetic energy and interaction energy for a function in a heavy interval on an interval length L_i with norm m_i is bounded below by

$$(1-\gamma)^2 \frac{m_i^2\pi^2}{L_i^2} + \frac{g\rho L m_i^4}{2L_i} \tag{13}$$

Ground State Bernoulli Case

Optimizing this quantity over $\sum m_i^2 = o(1)$, the kinetic energy must be less than the Lagrange parameter λ:

$$m_i^2 = \frac{L_i}{g\rho L}\left(\lambda - (1-\gamma)^2 \frac{\pi^2}{L_i^2}\right)$$

or

$$m_i^2 = 0$$

which means that the interval is not in M_{heavy}. With the normalization condition, this λ determines a minimal interval length for intervals in M_{heavy}. This minimal interval length is approximately $\tilde{L}$.

Ground State Bernoulli Case

The ground state energy is then bounded below by the minimal interaction energy of the ground state restricted to M_{heavy}. At most, there are approximately

$$(pqL) * g\rho \log_p(g\rho) * \left(\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)\right)$$

sites in M_{heavy}, which gives a lower bound of

$$E_0 \geq O\left(\frac{1}{(\log_p(g\rho))^2}\right) \quad (14)$$

Ground State Bernoulli Case

With probability one,

$$\limsup_{g\rho\to 0} \lim_{L\to\infty} \frac{E_0}{\frac{C_+}{\log_p^2(g\rho)}} = 1 \tag{15}$$

$$\liminf_{g\rho\to 0} \lim_{L\to\infty} \frac{E_0}{\frac{C_-}{\log_p^2(g\rho)}} = 1 \tag{16}$$

Conclusions

- Interaction changes the nature of the mean-field ground state and should not be treated as a perturbation in the large system limit.
- Localization is determined by

$$\mu(X_{>\epsilon}(\phi)) \geq \frac{gN(1-\epsilon^2)^2}{2E'} \tag{17}$$

- As seen in the Bernoulli case, random potentials are not a perturbation either. They fragment the space and in turn, the ground state.

Future Work

- Where are the phase transitions in terms of potential height b and interaction strength g? (Thomas-Fermi approximation)

- Is there a similar ‘delocalization’ criteria for the full tensor space?
- Is the mean field an appropriate approximation? What information does it provide on the true ground state?

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