Thermodynamic limit for interacting quantum particles in a random environment. General model

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Outline

- Physical motivation
- Mathematical definition and problem statement
- Existence theorems
- Ground state: perturbative approach

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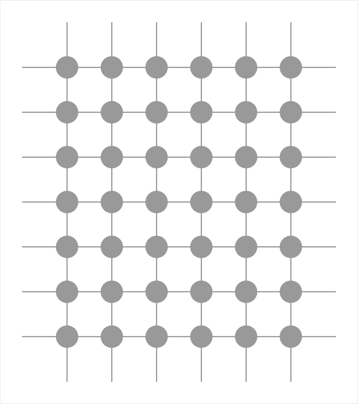
Interacting particles in disordered media



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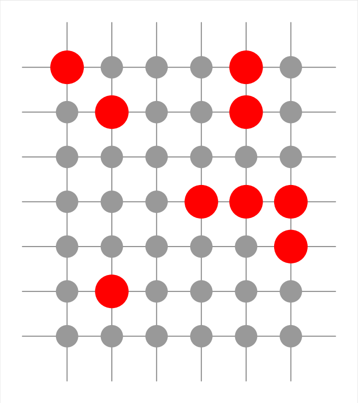

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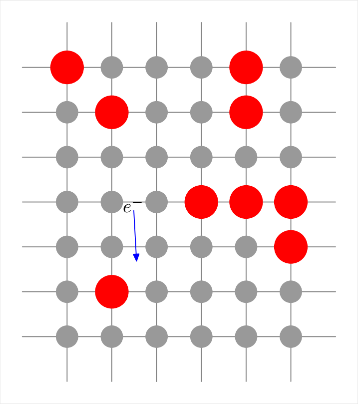

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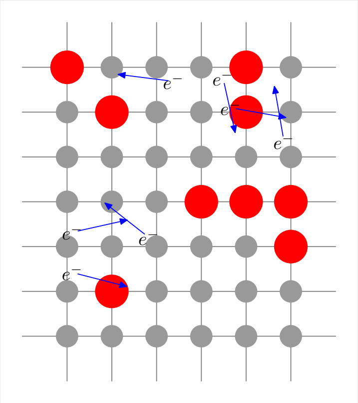

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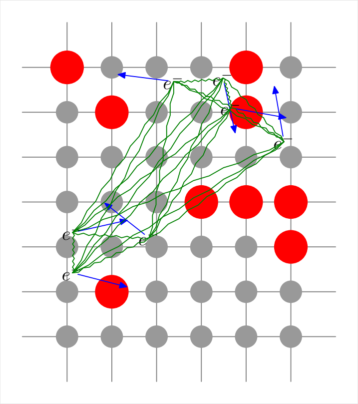

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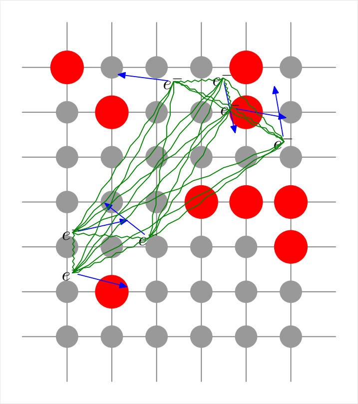

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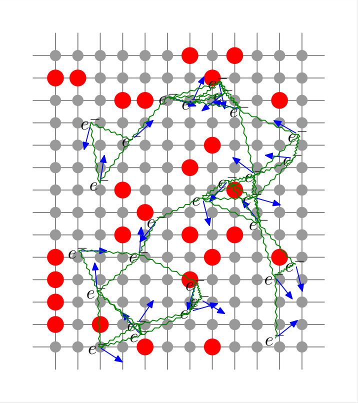

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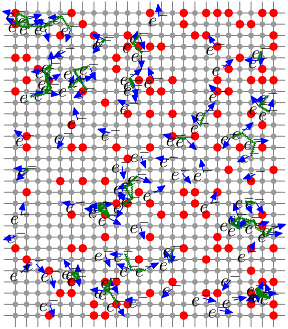

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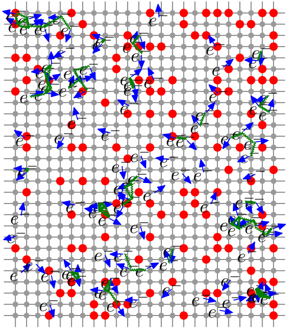

Problem

Describe a system of interacting quantum particles in the thermodynamic limit.

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Multiparticle systems in random environment

- One particle in random potential V_ω and domain $\Lambda \subset \mathbb{R}^d$:

$$H_\omega(\Lambda, 1) = -\Delta_d + V_\omega \quad \text{on} \quad \mathfrak{H}(\Lambda) = L^2(\Lambda)$$

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$$\mathfrak{H}^n(\Lambda) = \begin{cases} \bigotimes_{i=1}^{n} \mathfrak{H}(\Lambda) = L^2(\Lambda^n), & \text{Boltzmann statistics} \\ \bigotimes_{i=1}^{n}{}^{s} \mathfrak{H}(\Lambda) = L^2_{+}(\Lambda^n), & \text{Bose – Einstein statistics} \\ \bigwedge_{i=1}^{n} \mathfrak{H}(\Lambda) = L^2_{-}(\Lambda^n), & \text{Fermi – Dirac statistics} \end{cases}$$

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Thermodynamic limit and problem statement

- Large volume limit: $\Lambda \to \mathbb{R}^d$ if
  1. $|\Lambda| \to +\infty$,
  2. there exists a "shape" function π, satisfying $\lim_{\alpha\to 0} \pi(\alpha) = 0$, s.t. for α sufficiently small,

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- Problem:

describe $H^U_\omega(\Lambda, n)$ in the thermodynamic limit

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Conditions on a one-particle model

- $H_\omega(\Lambda, 1)$ is uniformly lower bounded and has a discrete spectrum:

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- Ergodicity: for $\gamma \in \mathbb{Z}^d$,

$$H(\Lambda + \gamma, 1) = U_\gamma H_{\tau_\gamma(\omega)} U_\gamma^*.$$

Conditions on interactions

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Example. Compactly supported interactions are tempered.

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Theorem

Suppose that

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2. *interactions are by pairs, translation invariant, stable and tempered.*

Then, there exists a function $\mathcal{E}^U(\rho)$ such that

$$\frac{E^U_\omega(\Lambda, n)}{n} \to \mathcal{E}^U(\rho), \quad \Lambda \to \mathbb{R}^d, \ \frac{n}{|\Lambda|} \to \rho > 0,$$

in L^2 with respect to ω. The energy density $\mathcal{E}^U$ is a non-random, increasing and convex in ρ^{-1}.

Energy subadditivity

- Main tool: subadditive type inequality. If

$$r := \mathrm{dist}(\Lambda_1, \Lambda_2) > R_0$$

then

$$E_\omega^U(\Lambda_1 \cup \Lambda_2, n_1 + n_2) \leqslant E_\omega^U(\Lambda_1, n_1) + E_\omega^U(\Lambda_2, n_2) + A n_1 n_2 r^{-\lambda}.$$

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- *Remark.* We strongly believe that for "reasonable" (but more general) U, the convergence is also ω-a.s.

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where the free part

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- First step: study the non-interacting system $H_\omega^0(\Lambda, n)$

Free particles: bosons and classical particles

Consider $H^0_\omega(\Lambda, n)$, i.e., take $U = 0$. We study $\Psi^0_\omega(\Lambda, n)$, $E^0_\omega(\Lambda, n)$ and $\mathcal{E}^0(\rho)$.

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Proposition

For Bose – Einstein and Maxwell – Boltzmann statistics, if $U = 0$, then

$$\mathcal{E}^0(\rho) = \inf \Sigma,$$

where Σ is the almost sure spectrum of $H_\omega(1)$.

Free particles: fermions

$$\Psi^0_\omega(\Lambda, n) = \frac{1}{\sqrt{n!}} \bigwedge_{j=1}^{n} \psi_j(\Lambda), \quad E^0_\omega(\Lambda, n) = \sum_{j=1}^{n} e_j(\Lambda)$$

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For Fermi – Dirac statistics, if $U = 0$*, then*

$$\mathcal{E}^0(\rho) = \frac{\int_{-\infty}^{E_\rho} E \mathrm{d}N(E)}{\int_{-\infty}^{E_\rho} \mathrm{d}N(E)}.$$

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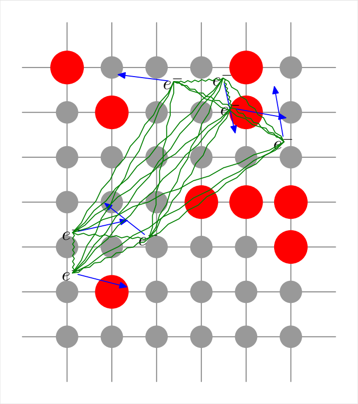

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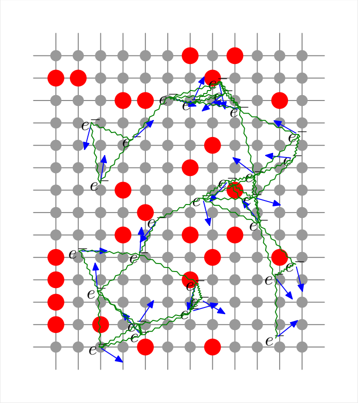

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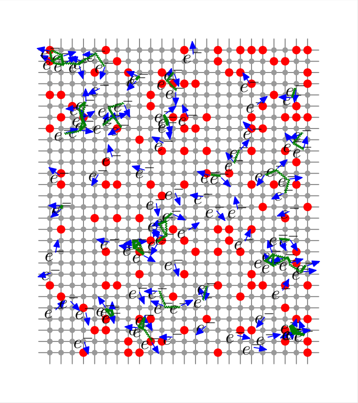

Acknowledgments

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