Mathematical Horizons
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Localization for Disordered Quantum Harmonic Oscillators

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based on joint work with

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Outline:

1. Introduction
2. Oscillator Models
3. Locality Estimates
4. Correlation Decay
5. Entanglement Bounds

Introduction

Overview

Over the past decade, there have been a number of important generalizations in the theory of:

- **Anderson Localization:** For a single particle, the techniques of multi-scale analysis and the fractional moment method have greatly improved.
- **Locality Estimates:** For many-body systems, there are new Lieb-Robinson bounds and correlation estimates for gapped models.

Question: Do these new results shed some light on localization for random many-body models?

Answer: Yes, at least for some very simple models. . .

Oscillator Models

Lattice Oscillators

Given the Hilbert Space

$$\mathcal{H}_L = \bigotimes_{x \in \Lambda_L} L^2(\mathbb{R}) \quad \text{where} \quad \Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$$

We will consider oscillator Hamiltonians

$$H_L = \sum_{x \in \Lambda_L} \left(\frac{1}{2m_x} p_x^2 + k_x q_x^2 \right) + \sum_{\langle x,y \rangle} \lambda_{x,y} (q_x - q_y)^2$$

Note: For each $x \in \Lambda_L$, the single site position and momentum are

$$q_x = \mathbb{1} \otimes \mathbb{1} \cdots \otimes q \otimes \cdots \otimes \mathbb{1} \quad \text{and} \quad p_x = \mathbb{1} \otimes \mathbb{1} \cdots \otimes i \frac{d}{dq} \otimes \cdots \otimes \mathbb{1}$$

both self-adjoint operators on $\mathcal{H}_L$.

Note: The real parameters: $\{m_x\}$, $\{k_x\}$, and $\{\lambda_{x,y}\}$ are resp. masses, spring constants, and interaction strengths

Observables and Dynamics

Denote by

$$\mathcal{A}_L = \mathcal{B}(\mathcal{H}_L) = \bigotimes_{x \in \Lambda_L} \mathcal{B}(L^2(\mathbb{R}))$$

an algebra of observables associated to Λ_L.

Example: For any $x \in \Lambda_L$ and $z \in \mathbb{C}$,

$$W_x(z) = \exp\left[i\left(\mathrm{Re}[z]q_x + \mathrm{Im}[z]p_x\right)\right] \in \mathcal{A}_L$$

is called a strictly local Weyl Operator with support $x \in \Lambda_L$.

The Heisenberg dynamics, or time evolution, associated to H_L is

$$\tau_t^L(A) = e^{itH_L} A e^{-itH_L} \text{ for any } A \in \mathcal{A}_L \text{ and } t \in \mathbb{R}$$

Basic Goal: Given parameters, understand the time evolution of local observables.

Locality Estimates

Deterministic Locality Bounds

A Lieb-Robinson bound for the constant coefficient case.

Theorem (Nachtergaele, Raz, Schlein, S. '09)

Let

$$m_x = m > 0, \quad k_x = k > 0, \quad \text{and} \quad \lambda_{x,y} = \lambda > 0$$

for all x, y. *There exist* $C < \infty$, $\eta > 0$, *and* $0 < v < \infty$ *such that*

$$\left\| \left[\tau_t^L(W_x(z)), W_y(z') \right] \right\| \leq C|z||z'|e^{-\eta(|x-y|-v|t|)}$$

for all L, x, y, z, *and* z'.

Note: If $x \neq y$, then $[W_x(z), W_y(z')] = 0$. The above shows that, in norm, the commutator is still small for $v|t| << |x-y|$.

v bounds the maximum velocity of propagation.

Disordered Oscillators

Consider the Hamiltonian

$$H_L = \sum_{x \in \Lambda_L} (p_x^2 + \mu k_x q_x^2) + \sum_{\langle x,y \rangle} (q_x - q_y)^2$$

where we set $2m_x = 1$, $\lambda_{x,y} = 1$, and

- Let $\{k_x\}_{x \in \mathbb{Z}^d}$ be an i.i.d. sequence of random variables with common distribution $d\mathbb{P}(k) = \rho(k)dk$ having $\rho \in L_c^\infty[0, \infty)$.
- Take $\mu > 0$ a disorder parameter.

Dynamical Localization

Theorem (Nachtergaele, S. , Stolz '12)

For the model above, assume that $\mu > 0$ is sufficiently large. There exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left[\tau_t^L(W_x(z)), W_y(z')\right] \right\| \right) \leq C|z||z'|e^{-\eta|x-y|}$$

for all L, x, y, z, and z'.

- This result holds uniformly in L.
- For $d = 1$, there is a similar result for arbitrary $\mu > 0$.

This is a strong form of dynamical localization. It establishes a zero-velocity Lieb-Robinson bound.

Effective One-Particle Hamiltonian

Crucial in the proof of this result is the fact that

$$H_L = (q^T, p^T) \begin{pmatrix} h_L & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

where

- $q = (q_x)_{x \in \Lambda_L}$ and $p = (p_x)_{x \in \Lambda_L}$ are regarded as vectors
- The effective one-particle Hamiltonian h_L satisfies

$$\langle f, h_L g \rangle = \sum_{<x,y>} \overline{(f(x) - f(y))}(g(x) - g(y)) + \sum_{x \in \Lambda_L} \mu k_x \overline{f(x)} g(x)$$

i.e. h_L is an Anderson model on $\ell^2(\Lambda_L)$.

Relating the Dynamics

The key estimate relating the many-body dynamics of H_L to the single-particle dynamics of h_L is:

$$\left\| \left[\tau_t^L(W_x(z)), W_y(z') \right] \right\| \leq |z||z'| \left\{ 2|\langle \delta_x, \cos(2th_L^{1/2})\delta_y \rangle| + \right.$$
$$\left. + |\langle \delta_x, h_L^{1/2} \sin(2th_L^{1/2})\delta_y \rangle| + |\langle \delta_x, h_L^{-1/2} \sin(2th_L^{1/2})\delta_y \rangle| \right\}$$

Technical Issue: Since the spring constants k_x have support in a neighborhood of 0, the operator $h_L^{-1/2}$ is unbounded.

Singular Eigenfunction Correlators

Lemma

Let h_L be the d-dimensional Anderson model at sufficiently large disorder. For every $\alpha > -1$, there exists $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} |\langle \delta_x, h_L^\alpha g(h_L)\delta_y\rangle|\right) \leq Ce^{-\eta|x-y|}$$

for all L and $x, y \in \Lambda_L$.

- For $\alpha = 0$, this is a well studied quantity in the mathematics community studying Anderson localization.
- The proof of this lemma uses a Riemann sum argument and known results on dynamical localization.
- Applying this result , e.g. with $\alpha = -1/2$ and $g_t(x) = \sin(2tx)$, completes the proof of the theorem.

Correlation Decay

On Deterministic Ground State Correlations

Let Ω_L be the non-degenerate, normalized ground state of H_L.

For any $A \in \mathcal{A}_L$, denote by

$$\langle A \rangle = \langle \Omega_L, A\Omega_L \rangle$$

the expected value of A in the ground state.

Results for deterministic gapped H_L:

Theorem (Cramer, Eisert '06; Cramer, Eisert, Serafini '07)

Let H_L be chosen with $\lambda_{x,y} = 1$, $2m_x = 1$, and $0 < a \leq k_x \leq b < \infty$ for all x, y. Then there exist $C < \infty$ and $\eta > 0$ such that

$$\left| \langle W_x(z) W_y(z') \rangle - \langle W_x(z) \rangle \langle W_y(z') \rangle \right| \leq C e^{-\eta |x-y|}$$

for all L, x, y, z, and z'.

Note: Lower bound on k_x ensures the model is gapped.

On Disordered Ground State Correlations

Theorem (Nachtergaele, S. , Stolz '12)

Let H_L be chosen with $\lambda_{x,y} = 1$, $2m_x = 1$, and $\{k_x\}$ i.i.d. with $d\mathbb{P}(k) = \rho(k)\,dk$ and $\rho \in L_c^\infty[0,\infty)$. Then there exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\left|\langle W_x(z)W_y(z')\rangle - \langle W_x(z)\rangle\langle W_y(z')\rangle\right|\right) \leq C|z||z'|e^{-\eta|x-y|}$$

for all L, x, y, z, and z'.

- This holds for any $d \geq 1$ and any disorder.
- This result applies to some gapless models.
- We have similar results for dynamically evolved correlations.

The Proof

Proof uses

$$\left|\langle W_x(z)W_y(z')\rangle - \langle W_x(z)\rangle\langle W_y(z')\rangle\right| \leq \frac{1}{2}|z||z'|\times$$
$$\times\left(|\langle\delta_x, h_L^{-1/2}\delta_y\rangle| + |\langle\delta_x, h_L^{1/2}\delta_y\rangle|\right)$$

and a contour integration cutting through the localized part of the spectrum.

Basic Idea: Despite the fact that the model is gapless, localization provides an effective mobility gap above the ground state.

Note again: These static results hold in any dimension and at any disorder. At large disorder, there are also results for dynamic correlations . . .

Results for Thermal States

We have similar results for correlations of thermal states.

For any $\beta > 0$, consider the thermal state

$$\mathcal{P}_{L,\beta} = \frac{e^{-\beta H_L}}{\mathrm{Tr}[e^{-\beta H_L}]}$$

For any observable $A \in \mathcal{A}_L$,

$$\langle A \rangle_\beta = \mathrm{Tr}[A\mathcal{P}_{L,\beta}]$$

is the expected value of A in the thermal state $\mathcal{P}_{L,\beta}$.

Theorem (Nachtergaele, S. , Stolz '12)

Let H_L be as above. Then there exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\left|\langle W_x(z)W_y(z')\rangle_\beta - \langle W_x(z)\rangle_\beta \langle W_y(z')\rangle_\beta\right|\right) \leq C|z|^{1/2}|z'|^{1/2}e^{-\eta|x-y|}$$

for all L, x, y, z, and z'.

Note: Also dynamical results at large disorder. . .

Entanglement Bounds

Entanglement Entropy

The Set-Up:

Fix a finite set $\Gamma \subset \mathbb{Z}^d$.

Take L large enough so that $\Gamma \subset \Lambda_L$.

Form a Bipartite decomposition: $\mathcal{H}_L = \mathcal{H}_1 \otimes \mathcal{H}_2$ with

$$\mathcal{H}_1 = \bigotimes_{x \in \Gamma} L^2(\mathbb{R}) \quad \text{and} \quad \mathcal{H}_2 = \bigotimes_{x \in \Lambda_L \setminus \Gamma} L^2(\mathbb{R})$$

Determine the ground state projector: $\mathcal{P}_L = |\Omega_L\rangle\langle\Omega_L|$.

Trace out the exterior degrees of freedom: $\mathcal{P}_L^1 = \mathrm{Tr}_{\mathcal{H}_2}[\mathcal{P}_L]$.

Calculate the Entanglement Entropy of this restriction:

$$S(\mathcal{P}_L^1) = -\mathrm{Tr}[\mathcal{P}_L^1 \ln(\mathcal{P}_L^1)]$$

Area Laws for Ground States

Theorem (Nachtergaele, S. , Stolz)

Let H_L be chosen with $\lambda_{x,y} = 1$, $2m_x = 1$, and $\{k_x\}$ i.i.d. with $d\mathbb{P}(k) = \rho(k)\,dk$ and $\rho \in L_c^\infty[0,\infty)$. Then there exist $C' < \infty$ such that

$$\mathbb{E}\left(S(\mathcal{P}_L^1)\right) \le C'|\partial\Gamma|$$

for all L with $\Gamma \subset \Lambda_L$.

- For the deterministic gapped case, such (surface) area laws are known e.g. Cramer, Dreissig, Eisert, Plenio '04,'05.
- Here the systems may be gapless, but we again exploit localization to achieve an effective mobility gap.

Back to Thermal States

We also have similar bounds that hold for thermal states.

Consider the same **Set-Up** as before, i.e., fix Γ, take L large with $\Gamma \subset \Lambda_L$, and write $\mathcal{H}_L = \mathcal{H}_1 \otimes \mathcal{H}_2$.

Define the logarithmic negativity of the thermal state $\mathcal{P}_{L,\beta}$ with respect to this bipartite decomposition by:

$$\mathcal{N}(\mathcal{P}_{L,\beta}) = \ln \left(\|\mathcal{P}_{L,\beta}^{T_1}\|_1 \right)$$

Here the partial transpose used above is defined by $(A \otimes B)^{T_1} = A^T \otimes B$ (where $A \mapsto A^T$ is any transposition in $\mathcal{H}_1$) and extended linearly to a larger class of observables.

Area Laws at Positive Temperature

Theorem (Nachtergaele, S. , Stolz '13)

Let H_L be chosen with $\lambda_{x,y} = 1$, $2m_x = 1$, and $\{k_x\}$ i.i.d. with $d\mathbb{P}(k) = \rho(k)\,dk$ and $\rho \in L_c^\infty[0,\infty)$. Then there exist $C' < \infty$ such that

$$\mathbb{E}\left(\mathcal{N}(\mathcal{P}_{L,\beta})\right) \leq C'|\partial\Gamma|$$

for all L with $\Gamma \subset \Lambda_L$.

Important Observation:

Lemma (Vidal-Werner '02)

If $\mathcal{P}$ is a rank-one projection on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{P}^1 = \mathrm{Tr}_{\mathcal{H}_2}[\mathcal{P}]$, then

$$S(\mathcal{P}^1) \leq \mathcal{N}(\mathcal{P})$$

We use this to prove our area law for the ground state.

Conclusions

For this simple random model, one can relate localization properties of the effective single-particle evolution to relevant many-body quantities.

The goal is to identify signatures of many-body localization, perhaps zero-velocity Lieb-Robinson bounds, decay of correlations, or area laws, which could possibly be established in more general random many-body systems.

References:

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