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The Bogoliubov c-Number Approximation for Random Boson Systems

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- **Problem 1:** To justify the Bogoliubov c-number approximation for the case of interacting Bose-gas in a *homogeneous random* media (**localised states**).
- **Problem 2:** Taking into account occurrence of generalised (*extended/fragmented*) Bose-Einstein condensation in an infinitesimal band of low **kinetic-energy modes**, to generalise the c-number substitution procedure for this band of low-momenta modes.

I. The Bogoliubov c-Number Approximation

1. Imperfect Bose-Gas

- Let interacting bosons of mass m be enclosed in a *cubic* box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ of the volume $V \equiv |\Lambda| = L^3$, with (*for simplicity*) periodic boundary conditions on $\partial\Lambda$: $t_\Lambda := (-\hbar^2 \Delta / 2m)_{p.b.c.}$
- $u(x)$ is *isotropic* two-body interaction with (*non-negative*):

$$v(q) = \int_{\mathbb{R}^3} d^3x \; u(x) e^{-iqx}, \; u \in \mathcal{L}^1(\mathbb{R}^3)$$

- The second-quantized Hamiltonian (of *imperfect* Bose-gas) acting *in* the boson Fock space $\mathfrak{F} := \mathfrak{F}_{boson}(\mathcal{H} = \mathcal{L}^2(\Lambda))$ is

$$H_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k b_k^* b_k + \frac{1}{2V} \sum_{k_1, k_2, q \in \Lambda^*} v(q) \, b_{k_1+q}^* b_{k_2-q}^* b_{k_2} b_{k_1}$$

where (*dual*) set $\Lambda^* = \left\{ k \in \mathbb{R}^3 : , \; k_\alpha = 2\pi n_\alpha / L \text{ et } n_\alpha \in \mathbb{Z}, \alpha = 1, 2, 3 \right\}$ and $\{\varepsilon_k\}_{k \in \Lambda^*} = \mathrm{Spec}(t_\Lambda)$.

• Here $\{\varepsilon_k = \hbar^2 k^2/2m \geq 0\}_{k\in\Lambda^*}$ is the one-particle excitations spectrum. The *perfect* Bose-gas Hamiltonian and *particle-number* operators are

$$T_\Lambda := \sum_{k\in\Lambda^*} \varepsilon_k \, b_k^* b_k \quad , \quad N_k := b_k^* b_k \ , \quad N_\Lambda := \sum_{k\in\Lambda^*} N_k$$

• $\left\{b_k^*, b_k\right\}_{k\in\Lambda^*}$ are boson creation and annihilation operators in the one-particle eigenstates verifying CCR $[b_k, b_q^*] = \delta_{k,q}$:

$$\psi_k\,(x) = \frac{1}{\sqrt{V}}\, e^{ikx}\, \chi_\Lambda(x) \in \mathcal{H},\ \ k \in \Lambda^*$$

$$b_k := b\,(\psi_k) = \int_\Lambda dx\ \overline{\psi_k}\,(x)\, b\,(x) \ \ , \ \ b_k^* = (b\,(\psi_k))^*$$

• $b^{\#}\,(x)$ are boson-field operators in the Fock space over $\mathcal{H}$.

1.3 Grand-Canonical (β,μ)-Ensemble

- (β,μ)-state generated by H_Λ on algebra $\mathfrak{A}(\mathfrak{F})$

$$\langle A\rangle_{H_\Lambda} := \mathrm{Tr}_{\mathfrak{F}}(e^{-\beta(H_\Lambda-\mu N_\Lambda)}\, A)/\mathrm{Tr}_{\mathfrak{F}} e^{-\beta(H_\Lambda-\mu N_\Lambda)}\ ,\ \ A\in\mathfrak{A}(\mathfrak{F})$$

- Grand-canonical pressure: $p[H_\Lambda](\beta,\mu) := (\beta V)^{-1}\ln \mathrm{Tr}_{\mathfrak{F}} e^{-\beta(H_\Lambda-\mu N_\Lambda)}$ for temperature β^{-1} and chemical potential μ.
- **Example:** For the *perfect* Bose-gas T_Λ one must put $\mu<0$, then expectation value of the particle number in mode k is

$$\langle b_k^* b_k\rangle_{T_\Lambda} := \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}\ ,\quad \varepsilon_k\geq 0\ .$$

- Expectation value of the *total* density of bosons in Λ:

$$\rho_\Lambda(\beta,\mu) := \frac{1}{V}\langle b_0^* b_0\rangle_{T_\Lambda} + \frac{1}{V}\sum_{k\in\Lambda^*\setminus\{0\}}\langle b_k^* b_k\rangle_{T_\Lambda} = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} + \rho_\Lambda(\beta,\mu)^*$$

- The critical density: $\rho_c(\beta) := \lim_{\mu\uparrow 0}\lim_{\Lambda\uparrow\mathbb{R}^3}\rho_\Lambda(\beta,\mu)^* < \infty$.

1.4 Conventional Bose-Einstein Condensation [F.London(1938)]

- For a fixed density ρ, let $\mu_\Lambda(\beta, \rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta, \mu) \quad \Rightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\rho)) \qquad (always\ exists!).$$

- *low* density : $\lim_\Lambda \ \mu_\Lambda(\rho < \rho_c(\beta)) = \mu_\Lambda(\rho) < 0$
- *high* density: $\lim_\Lambda \ \mu_\Lambda(\rho \geq \rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \ \left\{e^{-\beta\, \mu_\Lambda(\rho \geq \rho_c(\beta))} - 1\right\}^{-1} \Rightarrow$$

$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = -\frac{1}{V} \ \frac{1}{\beta(\rho - \rho_c(\beta))} + o(1/V) \ .$$

- Since $\varepsilon_k = \hbar^2 \sum_{j=1}^{d} (2\pi n_j / V^{1/3})^2 / 2m$, the BEC is in **k=0**-mode:

$$\lim_\Lambda \frac{1}{V} \left\{e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\rho))} - 1\right\}^{-1} = 0 \ ,$$

- This is a well-known *conventional* zero-mode (i.e.**type I**) BEC.

NB Generalised (Fragmented) Bose-Einstein Condensation

[Van den Berg-Lewis-Pulé (1982-86)] Let $\Lambda = L_1 \times L_2 \times L_3 = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$, $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

- **The Casimir box (1968):** Let $\alpha_1 = 1/2$, i.e. $\alpha_{2,3} < 1/2$.

Since $\varepsilon_{k_1,0,0} = \hbar^2(2\pi n_1/V^{1/2})^2/2m \sim 1/V$, then again the asymptotics of solution:

$$\rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\rho)) \Rightarrow \mu_\Lambda(\rho \geq \rho_c(\beta)) = -A/V + o(1/V), \ A \geq 0$$

$$\lim_\Lambda \left\{ \frac{1}{V} \frac{1}{e^{-\beta\mu_\Lambda(\rho)} - 1} + \frac{1}{V} \sum_{k \in \{\Lambda^* : n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\rho))} - 1} \right\}$$

$$= \rho - \rho_c(\beta) > 0 \ , \ \lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} \neq 0, \varepsilon_{k \neq 0} = \varepsilon_{k_1,0,0}$$

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0, \ \varepsilon_{0,k_{2,3} \neq 0} = \hbar^2(2\pi n_{2,3}/V^{\alpha_{2,3}})^2/2m$$

- Generalised BEC **type II** [van den Berg-Lewis-Pulé (1978)]:

$$\rho - \rho_c(\beta) = \lim_{L\to\infty} \frac{1}{V} \sum_{n_1 \in \mathbb{Z}} \left\{ e^{\beta(\hbar^2 (2\pi n_1/V^{1/2})^2/2m - \mu_\Lambda(\rho))} - 1 \right\}^{-1}$$

$$= \sum_{n_1 \in \mathbb{Z}} \frac{\beta^{-1}}{\hbar^2 (2\pi n_1)^2/2m + A} \Rightarrow \lim_{\Lambda} \frac{1}{V} \langle b_0^* b_0 \rangle_{T_\Lambda(\mu_\Lambda(\rho))} < \rho - \rho_c(\beta)!$$

Here $A \geq 0$ is a *unique root* of the above equation.

- **NB** For $\alpha_1 = 1/2$ the BEC is still mode by mode **macroscopic**, but it is **infinitely** *fragmented = quasi-condensate.*

Observed in experiments with *rotating* condensate (2000) and *chaotic* phases (2008).

- **The van den Berg box (1978):** $\alpha_1 > 1/2$.
- **Proposition** No macroscopic occupation of **any(!)** level:

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0.$$

• Generalised BEC **type III** [van den Berg-Lewis-Pulé (1978)]: $\alpha_1 > 1/2$ i.e. $\alpha_2 + \alpha_3 < 1/2$.

• Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{\alpha_1})^2/2 \;\sim\; 1/V^{2\alpha_1}$, $2\alpha_1 > 1$, then the solution $\mu_\Lambda(\rho)$ has **a new** asymptotics:

$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = -B/V^\delta + o(1/V^\delta),\; B \geq 0,\; \delta = 2\,(1-\alpha_1) =: 1-\epsilon$$

$$0 < \rho - \rho_c(\beta) = (2\pi\beta)^{-1/2} \int_0^\infty d\xi\; e^{-\beta B \xi}\; \xi^{-1/2}\;.$$

• The parameter $B = B\,(\beta,\rho) > 0$ is the *unique* root of the equation:

$$\rho - \rho_c(\beta) = \frac{1}{\sqrt{2\beta^2 B(\beta,\rho)}}\;.$$

- **Generalised BEC of type III**: one-mode particle occupations:

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\rho > \rho_c(\beta))) = 0 \text{ for } \mathbf{all} \ k \in \{\Lambda^*\} \ .$$

- For the *"renormalised"* k_1-modes occupation "density" one obtains:

$$\lim_{\Lambda} \frac{1}{V^{1-\epsilon}} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\rho > \rho_c(\beta))) = 2\beta \left(\rho - \rho_c(\beta)\right)^2 ,$$

where $k \in \{\Lambda^* : (n_1, 0, 0)\}$ and $1 - \epsilon = 1 - (2\alpha_1 - 1)$.

- **Definition**[*van den Berg-Lewis-Pulé*] (generalised BEC)

$$\rho - \rho_c(\beta) := \lim_{\eta \to +0} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| \le \eta\}} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta, \rho))} - 1 \right\}^{-1} \ .$$

- **Saturation ρ_m-PROBLEM**: [*van den Berg*] Is it possible that: $\rho_c \le \rho_m \le \infty$ such that **type III (or II)** $\to$ **type I**, for $\rho \ge \rho_m$? Yes! (BEC with the Second Critical Point [Beau-Z]).

1.5. Why the Zero-Mode c-Number Substitution ? The Bogoliubov Theory (1947)

• **The first Bogoliubov's ansatz:** If one expects that Bose-Einstein condensation, which occurs in the mode $k = 0$ for the perfect Bose-gas, **persists** for a *weak* two-body interaction $u(x)$, then one can to **truncate** $H_\Lambda \to H_\Lambda^B$ and to keep in H_Λ^B only the *most important* "condensate" terms, in which at least **two** operators b_0^*, b_0 are involved. This give the Bogoliubov Weakly Imperfect Bose-Gas (WIBG) Hamiltonian H_Λ^B.

• **The second Bogoliubov's ansatz:** Since for large volume the "condensate" operators $b_0^*/\sqrt{V}$, $b_0/\sqrt{V}$ *almost commute*: $[b_0/\sqrt{V}, b_0^*/\sqrt{V}] = 1/V$, one may use *substitutions*:

$$b_0/\sqrt{V} \to c \cdot \mathbb{I} \ , \ b_0^*/\sqrt{V} \to c^* \cdot \mathbb{I} \ , \quad c \in \mathbb{C} \ ,$$

in the truncated grand-canonical WIBG Hamiltonian $H_\Lambda^B(\mu) := H_\Lambda^B - \mu N_\Lambda \to H_\Lambda^B(c, \mu)$ to produce a diagonalizable bilinear form.

1.6 Zero-Mode c-Number Approximation

- For the periodic boundary conditions on $\partial\Lambda$, let $\mathfrak{F}_0 := \mathfrak{F}_{boson}(\mathcal{H}_0)$ be the boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ spanned by $\psi_{k=0}(x) = \chi_\Lambda(x)/\sqrt{V}$.
- Let $\mathfrak{F}_0' := \mathfrak{F}_{boson}\left(\mathcal{H}_0^\perp\right)$ be the Fock space constructed on the orthogonal complement $\mathcal{H}_0^\perp$. Then $\mathfrak{F}_{boson}(\mathcal{H}) = \mathfrak{F}_{boson}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp)$ is isomorphic to the *tensor product*:

$$\mathfrak{F}_{boson}\left(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp\right) \approx \mathfrak{F}_{boson}(\mathcal{H}_0) \otimes \mathfrak{F}_{boson}(\mathcal{H}_0^\perp) = \mathfrak{F}_0 \otimes \mathfrak{F}_0',$$

- For any complex number $c \in \mathbb{C}$ the coherent vector in $\mathfrak{F}_0$ is

$$\psi_{0\Lambda}(c) := e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V} c\right)^k (b_0^*)^k \,\Omega_0 = e^{(-V|c|^2/2 + \sqrt{V} c\, b_0^*)}\, \Omega_0 \,,$$

where Ω_0 is the vacuum of $\mathfrak{F}$. Notice that

$$\frac{b_0}{\sqrt{V}}\, \psi_{0\Lambda}(c) = c\, \psi_{0\Lambda}(c) \equiv c \cdot \mathbb{I}\, \psi_{0\Lambda}(c)\,.$$

• **Definition.** The *c-number* Bogoliubov approximation of the grand-canonical Hamiltonian ($N_\Lambda := \sum_{k\in\Lambda^*} b_k^* b_k := b_0^* b_0 + N'_\Lambda$)

$$H_\Lambda(\mu) := H_\Lambda - \mu N_\Lambda \ , \ \mathrm{dom}(H_\Lambda(\mu)) \subset \mathfrak{F} \approx \mathfrak{F}_{boson}(\mathcal{H}_0) \otimes \mathfrak{F}_{boson}(\mathcal{H}_0^\perp)$$

is a *self-adjoint operator* $H_\Lambda(c,\mu)$ defined in $\mathfrak{F}'_0 = \mathfrak{F}_{boson}\left(\mathcal{H}_0^\perp\right)$, for any fixed vector $\psi_{0\Lambda}(c)$, by the closable sesquilinear form:

$$\left(\psi'_1, H_\Lambda(c,\mu)\psi'_2\right)_{\mathfrak{F}'_0} \equiv \left(\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\, \psi_{0\Lambda}(c) \otimes \psi'_2\right)_{\mathfrak{F}} \ ,$$

for vectors $(\psi_{0\Lambda}(c) \otimes \psi'_{1,2}) \in$ *form-domain* of the operator $H_\Lambda(\mu)$.

• **Remark.** Since $(b_0/\sqrt{V})\ \psi_{0\Lambda}(c) = c \cdot \mathbb{I}\ \psi_{0\Lambda}(c)$, the *c-number* approximation is *equivalent* to *substitutions*:

$$b_0/\sqrt{V} \to c \cdot \mathbb{I} \ , \ b_0^*/\sqrt{V} \to c^* \cdot \mathbb{I}$$

in the Hamiltonian

$$H_\Lambda(\mu) \to H_\Lambda(c,\mu) =: H'_\Lambda(z) - \mu(|z|^2\, \mathbb{I} + N'_\Lambda) \ , \quad z := c\,\sqrt{V}.$$

1.7 Exactness of the c-Number Approximation

• **Definition.** The grand-canonical pressure for Hamiltonian $H_\Lambda(\mu)$ and for its *c-number* Bogoliubov approximation $H'_\Lambda(z,\mu)$, are defined by:

$$p_\Lambda(\mu) := \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathfrak{F}} \exp[-\beta H_\Lambda(\mu)]$$

$$p'_\Lambda(\mu) := \frac{1}{\beta V} \ln \int_{\mathbb{C}} d^2 z \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z,\mu)]$$

• **Proposition 1 (Variational Principle)** [Ginibre (1968), Lieb-Seiringer-Yngvason (2005)]

$$e^{\beta V p_\Lambda(\mu)} \geq \int_{\mathbb{C}} d^2 z \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z,\mu)] \geq$$

$$\sup_\zeta \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(\zeta,\mu)] =: e^{\beta V p_{\Lambda,max}(\mu)}$$

• **Proposition 2** [Lieb-Seiringer-Yngvason (2005)]

$$\lim_{\Lambda} p_\Lambda(\mu) = \lim_{\Lambda} p'_\Lambda(\mu) = \lim_{\Lambda} p_{\Lambda,max}(\mu) \ ,$$

with the *rate* of convergence:

$$0 \leq p_\Lambda(\mu) - p_{\Lambda,max}(\mu) \leq \mathcal{O}((\ln V)/V)$$

• [Ginibre (1968), Bru-Z (1999)] The *rate* of convergence proved by the *Approximating Hamiltonian Method*(AHM):

$$0 \leq p_\Lambda(\mu) - p_{\Lambda,max}(\mu) \leq \mathcal{O}(1/\sqrt{V}) \ .$$

NB Although in [Ginibre (1968)] and [Lieb-Seiringer-Yngvason (2005)] the use of *coherent states* is essential, the method of the last paper efficiently exploits the Peierls-Bogoliubov and Berezin-Lieb inequalities instead of the AHM. More flexible, it covers also the case of *infinitely* many k-modes, provided the card$\{k : k \in I_\Lambda \subset \Lambda^*\} < c\, V^{1-\gamma}$, $\gamma > 0$, and it gives more accurate estimates.

1.8 The c-Number Approximation for Ideal Bose-Gas

- c-number substitution in g.-c. Hamiltonian $T_\Lambda(\mu) := T_\Lambda - \mu N_\Lambda$

$$T_\Lambda(\mu) \to T_\Lambda(c,\mu) = \sum_{k\in\Lambda^*\setminus\{0\}} (\varepsilon_k - \mu) b_k^* b_k - V\mu|c|^2$$

- Pressures (note that $\mu < 0$ and $\varepsilon_{k=0} = 0$):

$$p[T_\Lambda(\mu)] = \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathfrak{F}} \exp[-\beta T_\Lambda(\mu)] = \frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1}$$

$$p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \sum_{k\in\Lambda^*\setminus\{0\}} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1} + \mu|c|^2$$

$$0 \le p[T_\Lambda(\mu)] - p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \ln(1 - e^{\beta\mu})^{-1} - \mu|c|^2 =: \Delta_\Lambda(c,\mu)$$

- Variational Principle: $\{c : \inf_c \lim_\Lambda \Delta_\Lambda(c,\mu)\} = \{c_*(\mu)\} \Rightarrow$
$c_*(\mu < 0) = 0 \vee (\mu c_*(\mu)) |_{\mu=0} = 0 \Rightarrow$ BEC density is not defined.

1.9 Gauge Invariance and the Bogoliubov Quasi-Averages

- Since $[H_\Lambda, N_\Lambda] = 0$ (*total particle number conservation law*),

$$H_\Lambda = e^{i\varphi N_\Lambda} H_\Lambda e^{-i\varphi N_\Lambda}\ ,\ U(\varphi) := e^{i\varphi N_\Lambda}\ ,$$

H_Λ is invariant w.r.t. gauge transformations $U(\varphi)$.

- **Corollary:** The grand-canonical expectation value:

$$\left\langle b_0/\sqrt{V}\right\rangle_{H_\Lambda} (\beta, \mu) = 0.$$

- Let $H_{\Lambda,\nu}(\mu) := H_\Lambda(\mu) - \sqrt{V}(\nu\ b_0^* + \nu^* b_0)$, $\nu \in \mathbb{C}$. Then

$$\left\langle \frac{b_0}{\sqrt{V}} \right\rangle_{H_{\Lambda,\nu}} (\beta, \mu) \neq 0\ ,\left\langle \frac{b_{k\neq 0}}{\sqrt{V}} \right\rangle_{H_{\Lambda,\nu}} (\beta, \mu) = 0\ .$$

- **Question**: $\lim_{\nu\to 0} \lim_\Lambda \left\langle \frac{b_0}{\sqrt{V}} \right\rangle_{H_{\Lambda,\nu}} (\beta, \mu) =: c_0 \neq 0$?

Yes ! It is a spontaneous breaking of the gauge symmetry

- Here c_0 is the Bogoliubov quasi-average, [Bogoliubov (1960)].

Example: Ideal Bose-Gas

- Gauge breaking sources: $T_{\Lambda,\nu}(\mu) := T_{\Lambda}(\mu) - \sqrt{V}(\nu\, b_0^* + \nu^* b_0) =$

$$-\mu(b_0^* + \sqrt{V}\overline{\nu}/\mu)(b_0 + \sqrt{V}\nu/\mu) + T_{\Lambda}^{(k\neq 0)}(\mu) + V|\nu|^2/\mu.$$

- c-number substitution: $T_{\Lambda,\nu}(\mu) \to$

$$T_{\Lambda,\nu}(c,\mu) = -\mu V(\overline{c} + \overline{\nu}/\mu)(c + \nu/\mu) + T_{\Lambda}^{(k\neq 0)}(\mu) + V|\nu|^2/\mu$$

- Pressures (note that $\mu < 0$ and $\varepsilon_{k=0} = 0$):

$$p[T_{\Lambda,\nu}(\mu)] = p[T_{\Lambda}(\mu)] - |\nu|^2/\mu$$

$$p[T_{\Lambda,\nu}(c,\mu)] = p[T_{\Lambda}^{(k\neq 0)}(\mu)] + \mu V(\overline{c} + \overline{\nu}/\mu)(c + \nu/\mu) - |\nu|^2/\mu$$

$$0 \leq p[T_{\Lambda,\nu}(\mu)] - p[T_{\Lambda,\nu}(c,\mu)] =$$

$$\frac{1}{\beta V}\ln(1 - e^{\beta\mu})^{-1} - \mu|c + \eta/\mu|^2 =: \Delta_{\Lambda,\nu}(c,\mu)$$

- Variational Principle: $\{c : \inf_c \lim_{\Lambda} \Delta_{\Lambda,\nu}(c,\mu)\} = \{c_*(\mu,\nu) = -\nu/\mu\} \Rightarrow$

variational BEC density ρ_{0*} is defined by $|\nu/\mu(\nu)| \underset{\nu\to 0}{\longrightarrow} \sqrt{\rho_{0*}}$:

$$\rho_{0*} := \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} |c_*(\mu,\nu)|^2 \overset{!}{=} \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} \lim_{V\to\infty} \langle \frac{b_0^*}{\sqrt{V}} \rangle_{T_{\Lambda,\nu}(\mu)} \langle \frac{b_0}{\sqrt{V}} \rangle_{T_{\Lambda,\nu}(\mu)}$$

- BEC versus quasi-average BEC and maximizer ρ_{0*}

$$zero-mode\ BEC\ \rho_0 \Rightarrow \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu=0}(\mu)} = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} \leq$$

$$\frac{|\nu|^2}{\mu^2} + \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} = \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu}(\mu)} \Rightarrow quasi-average\ BEC$$

and by Variational Principle for the c-Number Approximation:

$$\lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} \lim_{V\to\infty} \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu}(\mu)} \overset{!}{=} \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} \lim_{V\to\infty} \langle \frac{b_0^*}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)} \langle \frac{b_0}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)}$$

$$\Rightarrow gauge-symmetry\ breaking\ BEC \overset{!}{=} \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} |c_*(\mu,\nu)|^2 = \rho_{0*}$$

- **Question**: Is it possible that $\rho_0 < \rho_{0*}$?

Yes ! For *ideal* as well as for *interacting* Bose-gas [Buffet-de Smedt-Pulé (1983)] with *generalised* BEC of type II/III.

• **Theorem** [Lieb-Seiringer-Yngvason (2005),(2007)]
($k = 0$ mode) BEC $\Rightarrow$ quasi-average BEC $\Leftrightarrow$ spontaneous gauge-symmetry breaking BEC $\Leftrightarrow$ non-zero c-number approximation for the mode $k = 0$

The proof is based on Griffith's arguments and on Propositions:

• **Proposition 3** For real ν

$$\lim_{\Lambda} p_\Lambda(\mu;\nu) = \lim_{\Lambda} p'_\Lambda(\mu;\nu) = \lim_{\Lambda} p_{\Lambda,max}(\mu;\nu)$$

and are convex in ν.

• **Proposition 4** (Gauge-Symmetry Breaking BEC)

$$\lim_{|\nu|\to 0,\ \arg(\nu)} \lim_{\Lambda} \left\langle \frac{b_0}{\sqrt{V}} \right\rangle_{H_{\Lambda,\nu}} (\beta,\mu) =$$

$$\lim_{|\nu|\to 0,\ \arg(\nu)} \lim_{\Lambda} |z_{\Lambda,max}(\nu)|\ e^{i\ \arg(\nu)}/\sqrt{V} =: c_0\ .$$

- Variational Principle: $z_{\Lambda,max}(\nu) = |z_{\Lambda,max}(\nu)|\ e^{i\ \text{arg}(\nu)}$,

$$\sup_{\zeta} \text{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(\zeta,\mu;\nu)] = \text{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z_{\Lambda,max}(\nu),\mu;\nu)]$$

$$= \exp\ [\beta V p_{\Lambda,z_{\Lambda,max}(\nu)}(\mu;\nu)] =: \exp\ [\beta V p_{\Lambda,max}(\mu;\nu)],$$

and $z_{\Lambda,max}(0) = |z_{\Lambda,max}(0)|\ e^{i\phi}$, $p_{\Lambda,z_{\Lambda,max}(\nu)}(\mu;\nu)|_{\nu=0} = p_{\Lambda,max}(\mu)$.

- **Corollary.** The quasi-average condensate density and the condensate density equation:

$$\rho_0(\beta,\mu) = \lim_{|\nu|\to 0,\ \text{arg}(\nu)} \lim_{\Lambda} \left\langle \frac{b_0^* \, b_0}{V} \right\rangle_{H_{\Lambda,\nu}} (\beta,\mu) = \lim_{\Lambda} |c_{0,\Lambda,max}|^2(\beta,\mu)\ .$$

where $c_{0,\Lambda,max}$ is a maximizer of the *variational problem*:

$$\sup_{c_0} \text{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(c_0\sqrt{V},\mu)] = \text{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(c_{0,\Lambda,max}\sqrt{V},\mu)]$$

II Random Homogeneous (Ergodic) External Potentials. Can we save the Bogoliubov Theory (BT)?

2.1 Random Eigenfunctions/Kinetic-Energy Eigenfunctions

- For *a.s.* self-adjoint random Schrödinger operator in $\Lambda \subset \mathbb{R}^d$:

$$h^{\omega}_{\Lambda}\,\phi^{\omega}_{j} = (t_{\Lambda} + v^{\omega})_{\Lambda}\,\phi^{\omega}_{j} = E^{\omega}_{j}\,\phi^{\omega}_{j}\ ,\ \text{for } a.a.\ \omega \in \Omega\ .$$

- Let $N_{\Lambda}(\phi^{\omega}_{j})$ be particle-number operator in the eigenstate ϕ^{ω}_{j}.

$$N_{\Lambda} := \sum_{j \geq 1} N_{\Lambda}(\phi^{\omega}_{j}) := \sum_{j \geq 1} b^{*}(\phi^{\omega}_{j}) b(\phi^{\omega}_{j})$$

is the *total* number operator in the boson Fock space $\mathfrak{F}(\mathcal{L}^2(\Lambda))$, $b(\phi^{\omega}_{j}) := \int_{\Lambda} dx\, \overline{\phi^{\omega}_{j}}(x)\, b(x)$, and $\{\phi^{\omega}_{j}\}_{j \geq 1}$ is a basis in $\mathcal{H} = \mathcal{L}^2(\Lambda)$.

- Let $t_{\Lambda}\,\psi_k = \varepsilon_k\,\psi_k$ be the kinetic-energy operator, eigenfunctions and eigenvalues $\varepsilon_k = \hbar^2\,k^2/2m$. A key hypothesis of the conventional Bogoliubov Theory is the existence of translation-invariant ground-state (or zero-mode $\psi_{k=0}$) condensation.

- Random Hamiltonian H^{ω}_{Λ} of interacting Bosons in $\mathfrak{F}(\mathcal{H})$:

$H^{\omega}_{\Lambda} := T^{\omega}_{\Lambda} + U_{\Lambda}$, random Schrodinger operator + interaction

$$\mathrm{d}\Gamma(h^{\omega}_{\Lambda}) := T^{\omega}_{\Lambda} = \sum_{j\geq 1} E^{\omega}_{j}\, b^{*}(\phi^{\omega}_{j}) b(\phi^{\omega}_{j}) = \sum_{k_1,k_2\in\Lambda^*} (\psi_{k_1}, (t_{\Lambda} + v^{\omega})\psi_{k_2})_{\mathcal{H}} b^{*}_{k_1} b_{k_2}$$

Two faces of the second-quantised $u(x-y)$ interaction in $\mathfrak{F}(\mathcal{H})$:

$$U_{\Lambda} := \frac{1}{2} \sum_{\substack{j_1,j_2\\ j_3,j_4}} (\phi^{\omega}_{j_1} \otimes \phi^{\omega}_{j_2}, u\ \phi^{\omega}_{j_3} \otimes \phi^{\omega}_{j_4})_{\mathcal{H}\otimes\mathcal{H}}\ b^{*}(\phi^{\omega}_{j_1}) b^{*}(\phi^{\omega}_{j_2})\ b(\phi^{\omega}_{j_3}) b(\phi^{\omega}_{j_4})$$

$$= \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v\,(q)\, b^{*}_{k_1+q} b^{*}_{k_2-q} b_{k_2} b_{k_1}$$

- The aim: Exactness of the Bogoliubov c-Number Approximation (definition ?) For example:

$\sum_{j:E^{\omega}_{j}\leqslant\delta} \langle N_{\Lambda}(\phi^{\omega}_{j})\rangle_{H^{\omega}_{\Lambda}}/V \to c$, **or** $\sum_{k:\varepsilon_k\leqslant\gamma} \langle N_{\Lambda}(\psi_k)\rangle_{H^{\omega}_{\Lambda}}/V \to c$ **?**

2.2 Random versus Kinetic-Energy Condensation

- **Theorem 2.1**[Jaeck-Pulé-Z(2009)]

Let $H^{\omega}_{\Lambda} := T^{\omega}_{\Lambda} + U_{\Lambda}$ be *many-body* Hamiltonian of interacting bosons in random external potential V^{ω}_{Λ}. If the particle *interaction U_{Λ}* commutes with **any** of number operators $N_{\Lambda}(\phi^{\omega}_{j})$ (*local gauge invariance*), then

$$a.s. - \lim_{\delta\downarrow 0} \lim_{\Lambda} \sum_{j: E^{\omega}_{j} \leqslant \delta} \frac{1}{V} \langle (N_{\Lambda}(\phi^{\omega}_{j}) \rangle_{H^{\omega}_{\Lambda}} > 0 \Leftrightarrow$$

$$\Leftrightarrow a.s. - \lim_{\gamma\downarrow 0} \liminf_{\Lambda} \sum_{k: \varepsilon_{k} \leqslant \gamma} \frac{1}{V} \langle N_{\Lambda}(\psi_{k}) \rangle_{H^{\omega}_{\Lambda}} > 0 \ ,$$

and: $\lim_{\gamma\downarrow 0} \lim_{\Lambda} \sum_{k: \varepsilon_{k} > \gamma} \langle N_{\Lambda}(\psi_{k}) \rangle_{H^{\omega}_{\Lambda}} / V = 0$. Here $\langle - \rangle_{H^{\omega}_{\Lambda}}$ is quantum Gibbs expectation with random Hamiltonian H^{ω}_{Λ}.

• **Remark 2.2** If a many-body interaction satisfies the "local" gauge invariance:

$$[U_\Lambda, N_\Lambda(\phi_j)] = 0 \ ,$$

then U_Λ is a **function** of the **occupation number operators** $\{N_\Lambda(\phi_j)\}_{j\geq 1}$. For this reason it is called a "*diagonal interaction*".

• **Corollary 2.3** The *random* localised generalised (type ?) boson condensation occurs **if and only if** there is a generalised (type II/III) condensation in the extended (*kinetic-energy*) eigenstates. This is **a possible way** to save the Bogoliubov theory in a *non-translation invariant*, but *homogeneous* random external potential.

2.3 Amounts of Random and Kinetic-Energy Condensates

• Let for any $A \subset \mathbb{R}_+$ the particle occupation measures m_Λ and $\tilde{m}_\Lambda$ are defined for the perfect Bose-gas by:

$$m_\Lambda(A) := \frac{1}{V} \sum_{j:E_i \in A} \langle N_\Lambda(\phi_i^\omega) \rangle_{T_\Lambda^\omega} \ , \ \tilde{m}_\Lambda(A) := \frac{1}{V} \sum_{k:\varepsilon_k \in A} \langle N_\Lambda(\psi_k) \rangle_{T_\Lambda^\omega} \ .$$

• **Theorem 2.4**[Jaeck-Pulé-Z(2009)] For *perfect* Bose-gas amounts of condensates coincide:

$$m(dE) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(dE) + (e^{\beta E} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} \geqslant \rho_c \ , \\ (e^{\beta(E-\mu_\infty)} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} < \rho_c \ , \end{cases}$$

$$\tilde{m}(d\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(d\varepsilon) + F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} \geqslant \rho_c, \\ F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} < \rho_c. \end{cases}$$

with **explicitly** defined density $F(\varepsilon)$. For models with *diagonal* interactions: $m_\Lambda(A) \leq \tilde{m}_\Lambda(A)$.

2.4 Example: BEC in One-Dimensional Random Potential. Poisson Point-Impurities

- For $d = 1$ *Poisson point-impurities*, $a > 0$:

$$v^{\omega}(x) := \int_{\mathbb{R}^1} \mu_{\tau}^{\omega}(dy) a\, \delta(x-y) = \sum_j a\, \delta(x - y_j^{\omega})$$

$$\mathbb{P}\{\omega : \mu_{\tau}^{\omega}(\Lambda) = s\} = \frac{|\Lambda|^s}{s!} e^{-\tau|\Lambda|} \ , \ \mathbb{E}(\mu_{\tau}^{\omega}(\Lambda)) = \tau|\Lambda| \ , \ \Lambda \subset \mathbb{R}^1 \ .$$

Proposition 2.5 Let $a = +\infty$. Then $\sigma(h^{\omega})$ is a.s. nonrandom, dense *pure-point* spectrum $\overline{\sigma_{p.p.}(h^{\omega})} = [0, +\infty)$, with IDS

$$\mathcal{N}(E) = \tau \frac{e^{-\pi\tau/\sqrt{2E}}}{1 - e^{-\pi\tau/\sqrt{2E}}} \sim \tau e^{-\pi\tau/\sqrt{2E}} , \ E \downarrow 0, \text{(Lifshitz tail).}$$

- **Spectrum:**

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{\pi^2 s^2 / 2(L_j^\omega)^2\right\}_{s=1}^{\infty}$$

- Intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are *i.i.d.r.v.* :

$$dP_{\tau, j_1, \ldots, j_k}(L_{j_1}, \ldots, L_{j_k}) = \tau^k \prod_{s=1}^{k} e^{-\tau L_{j_s}} dL_{j_s}$$

- **Eigenfunctions:**

For almost all $\omega \in \Omega$ the one-particle **localised** quantum states $\{\phi_j^\omega\}_{j \geq 1}$, give a basis in $L^2(\Lambda)$.

III.Generalized Bogoliubov c-numbers approximation

3.1 Existence of the approximating pressure

• Since randomness implies *fragmented* (or generalized type II/III) condensation, following the Bogoliubov approximation philosophy, we want to replace all creation/annihilation operators in the momentum states ψ_k with kinetic energy *less* than some $\delta > 0$ by c-numbers. Let $I_\delta \subset \Lambda^*$ be the set of all *replaceable* modes

$$I_\delta := \left\{ k \in \Lambda^* : \hbar^2 k^2 / 2m \leqslant \delta \right\},$$

and we denote $n_\delta := \mathrm{card}\{k : k \in I_\delta\}$.

• **Problem:** The number of quantum states n_δ is of the **order** V_l, since by definition of the Integrated Density of States: $n_\delta = V\ \mathcal{N}_\Lambda(\delta)$. To use the *Lieb-Seiringer-Yngvason method* we consider $n_{\delta_\Lambda} = O(V^{1-\gamma})$, $0 < \gamma < 1$. Why it is possible ?

• **Generalised BEC of type III**: one-mode particle occupations.

Definition [Jaeck-Pulé-Z(2010)] We call $\{\phi_j^\omega\}_{j\geq 1}$ localised if

$$\lim_{\Lambda} \frac{1}{\sqrt{V}} \int_{\Lambda} dx \; |\phi_j^\omega(x)| = 0 \quad \text{for a.a. } \omega$$

Theorem 3.1[Jaeck-Pulé-Z(2010), Jaeck-Z(2010)] Let all $\{\phi_j^\omega\}_{j\geq 1}$ be localised. Then for models H_Λ^ω with *diagonal interactions*

$$\lim_{\Lambda} \frac{1}{V} \langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} = 0 \text{ for } \textbf{all } k \in \{\Lambda^*\}$$

This implies that any possible *kinetic* generalised BEC in these models is of **type III**.

Therefore, the number of "condensed" kinetic-modes is at most $O(V^{1-\gamma})$, $0 < \gamma < 1$, and one can use the LSY method for modes:

$$\lim_{\Lambda} \frac{1}{V^\gamma} \langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} \neq 0 \;, \; \text{ for k} \in \text{I}_{\delta_\Lambda}, \; \gamma = 1 - \epsilon$$

• Let $\mathcal{H}^\delta$ be the subspace of $\mathcal{H}$ spanned by the set of ψ_k with $k \in I_\delta$, and P_δ be orthogonal projector onto this subspace. Hence, we have a natural decomposition of the total space $\mathcal{H}$ and the corresponding representation for the associated symmetrised Fock space:

$$\mathcal{H} = \mathcal{H}^\delta \oplus \mathcal{H}' \ , \qquad \mathfrak{F} \approx \mathfrak{F}^\delta \otimes \mathfrak{F}'.$$

• Then we proceed with the Bogoliubov substitution $b_k \to c_k$ and $b_k^* \to \bar{c}_k$ for all $k \in I_\delta$, which provides an *approximating* (for the initial) Hamiltonian, that we denote by $H_\Lambda^{low}(\mu, \{c_k\})$.

• The partition function and the corresponding pressure for this approximating Hamiltonian have the form:

$$\Xi_{\Lambda}^{low}(\mu, \{c_k\}) = \mathrm{Tr}_{\mathfrak{F}'}\, e^{-\beta H_{\Lambda}^{low}(\mu, \{c_k\})} \ ,$$

$$p_{\Lambda,\delta}^{low}(\mu, \{c_k\}) = \frac{1}{V} \ln \Xi_{\Lambda}^{low}(\mu, \{c_k\}) \ .$$

• **Theorem 3.2 [Jaeck-Z]** The c-numbers substitution for all operators in the energy-band I_{δ_Λ}, card$\{k : k \in I_{\delta_\Lambda}\} = O(V^{1-\gamma})$, does not affect the original pressure in the following sense:

$$\text{a.s.}-\lim_{\Lambda}[p_\Lambda(\beta,\mu) - \{\max_{\{c_k\}} p^{low}_{\Lambda,\delta_\Lambda}(\mu,\{c_k\})\}] = 0$$

• **Remark 3.1** The one-mode case: [Ginibre,Buffet-de Smedt-Pulé, Lieb-Seiringer-Yngvason].

• **Remark 3.2** For eventual *type II* condensation the arguments are similar with a volume dependent cut-off of the converging sum over modes.

THANK YOU FOR YOUR ATTENTION !

0. Motivation and a Brief History

• One of the key developments in the theory of the Bose gas, especially the theory of the low density gases currently at the forefront of experiment, is Bogoliubov's 1947 analysis of the many-body Hamiltonian by means of a c-number substitution for the most relevant operators in the problem, the zero-momentum mode operators, namely $b_0 \to z$, $b_0^* \to z^*$. Later this idea triggered a more general Approximating Hamiltonian Method [Bogoliubov (1960), Bogoliubov (jr)(1965), Brankov, Tonchev and Z(1974)].

• Naturally, the appropriate value of z has to be determined by some sort of consistency or variational principle, which might be complicated, but the concern is whether this sort of substitution is legitimate, i.e., error free.

- The rigorous justification for this substitution, as far as calculating the pressure of interacting (superstable) boson gas is concerned, was done in the paper [Ginibre (1968)] and revised and essentially improved by [Lieb-Seiringer-Yngvason(2005), (2007)]
- The rigorous justification for this *substitution*, as far as calculating the pressure is concerned, was done in [Ginibre (1968)]. In textbooks it is often said, for instance, that it is tied to the imputed "fact" that the expectation value of the number operator $n_0 = b_0^* b_0$ is of order V = volume. This was the (second) Bogoliubov ansatz: Bose-Einstein condensation (BEC) *justifies the substitution*.
- As Ginibre pointed out, however, BEC has nothing to do with it. The z substitution still gives the right answer for any value of the Gibbs average of n_0.