BSDE, PDE, Nonlinear Expectation and Model Uncertainty

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Abstract. A solution of a linear BSDE (Backward Stochastic Differential Equation) is a discounted martingale with a prescribed terminal value under a specific probability called martingale measure through a Girsanov transformation. In fact a general nonlinear BSDE, though very different from an SDE, can be still solved by a fixed point approach. It turns out that the BSDE can be considered as a nonlinear Girsanov transformation and that the solution of the BSDE is a nonlinear martingale under a nonlinear expectation, called g-expectation. The corresponding nonlinear Feynman-Kac formula tells us that, once the coefficients depend only on the state of the Brownian path, then the BSDE is a quasilinear PDE of parabolic type. This reveals that, a general (non-Markovian) BSDE is in fact a PDE in which the Brownian path plays the role of state variable x. The g-expectation then is the nonlinear semigroup associated with the PDE. The above Brownian motion can also be replaced by a general and possibly degenerate Markovian process.

For a fully nonlinear parabolic PDE, can one still establish the corresponding BSDE, nonlinear expectation and path-dependence PDE? The answer of this deep problem is: the Wiener probability measure corresponding to the Brownian motion can no longer be the reference probability space of the corresponding nonlinear expectation since the later is 'fully nonlinear' which cannot be 'absolutely continuous' w. r. t. this measure. A direct solution to this problem is to use the PDE to construct the corresponding nonlinear expectation, called G-expectation. The nonlinear martingale under this expectation, called G-martingale, is the solution of the BSDE which can be regarded as the corresponding fully nonlinear path-dependence PDE. It turns out that a well-designed G-expectation can be used to dominate a large set of fully nonlinear expectations, or BSDE. In this framework, the corresponding canonic process is called G-Brownian motion which is a continuous process with independent and stable increments. The increments are proved to be G-normal distributed which coincides with the limit distribution of the central limit theorem under a nonlinear expectation. This approach also gives us a pedagogically direct and simple access to the theory of G-expectation, G-Brownian motion and the corresponding stochastic calculus of Itô's type. BSDE, nonlinear expectations (G-Expectations) and the corresponding G-martingales are useful tools to solve risk measure problem for situations under probability and/or distribution uncertainties.

Course 1. Preliminary;
Course 2. BSDE, FBSDE and Nonlinear Feynman-Kac formula;
Course 3. Nonlinear expectations and through PDE;
Course 4. Financial risk under Knightian uncertainty.