A computable absolutely normal Liouville number

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Theorem (Becher, Heiber, Slaman, 2014)

There is a computable absolutely normal Liouville number.

Normal numbers

A *base* is an integer b greater than or equal to 2.

Definition (Borel, 1909)

A real x is simply normal to base b if in the expansion of x in base b, each digit occurs with limiting frequency equal to $1/b$.

A real x is normal to base b if x is simply normal to every base b^j, for every positive integer j.

A real x is absolutely normal if x is normal to every base.

Equivalently, x is normal to base b if every block of digits occurs in the expansion of x in base b with limiting frequency equal to $1/b^j$, where j is the block length.

Existence of absolutely normal numbers

In 1909 Borel proved that the set of absolutely normal numbers has Lebesgue measure one. He asked for one example.

Constructions of absolutely normal numbers (1917-2013) accounted for no other mathematical (geometric, algebraic, number theoretic) properties.

Uniform distribution modulo one

Let $\{x\}$ be the fractional part of a real x.

A sequence of reals $(x_j)_{j\geq 1}$ is uniformly distributed modulo one (u.d. mod 1), if for every subinterval I of the unit interval,

$$\lim_{n\to\infty} \frac{\#\{j : 1 \leq j \leq n \text{ and } \{x_j\} \in I\}}{n} = |I|.$$

Discrepancy

Discrepancy of finite sequence with respect to a fixed I is

$$D(I, (\{x_j\} : 1 \leq j \leq n)) = \left| \frac{\#\{j : 1 \leq j \leq n \text{ and } \{x_j\} \in I\}}{n} - |I| \right|.$$

Discrepancy with respect to every subinterval in the unit interval:

$$D(\{x_j\} : 1 \leq j \leq n) = \sup_{I \subseteq [0,1]} D(I, (x_j : 1 \leq j \leq n)).$$

Thus, $(x_j)_{j \geq 1}$ is u.d. mod 1 if $\lim_{n \to \infty} D(\{x_j\} : 1 \leq j \leq n) = 0$.

Observation

Discrepancy with respect to subintervals is continuous in the first argument.

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if, and only if, $(b^j x)_{j\geq 0}$ is u.d. mod 1.

Thus, a real x is normal to base b exactly when $\lim_{n\to\infty} D(\{b^j x\} : 0 \leq j < n) = 0$.

Observation

Let b be a base, let I be an interval. For all sufficiently large n, if x and y are real numbers such that $|x - y| < b^{-n}$.

$$D(I, \{b^j x\} : 0 \leq j < n) \approx D(I, \{b^j y\} : 0 \leq j < n).$$

Thus, discrepancy of normality is continuous in the second argument.

Construction of a computable absolutely normal number

Consider a computable $(I)_{n\geq 1}$ of intervals $I_n = [x_n, x_n + 2^{-a_n})$ where a_n is a positive integer and x_n is dyadic rational with precision a_n, such that

- $I_1 = [0, 1)$ and for all $n \geq 1$, $I_{n+1} \subset I_n$,
- $\lim_{n\to\infty} |I_n| = 0$,
- for every base b, $\lim_{n\to\infty} D(\{b^j x_n\} : 0 \leq j < \langle a_n; b\rangle) = 0$, where $\langle a; b\rangle$ denotes $\lceil a/ \log_2 b\rceil$.

Thus, the dyadic rational x_{n+1} is obtained at step $n+1$ by concatenating the expansion of x_n with a new block of digits.

For every base $b \leq n+1$, $D(\{b^j x_{n+1}\} : \langle a_n; b\rangle \leq j < \langle a_{n+1}; b\rangle)$ is small.

Then the real $x = \bigcap_{i\geq 1} I_i = \lim_{n\to\infty} x_n$ is well defined and, by continuity of discrepancy, x is absolutely normal.

Weyl's criterion

A sequence is u.d. in the unit interval if for any Riemann integrable f, $\int_0^1 f(x)dx$ is the limit of the average values of f in the points of the sequence.

Theorem (Weyl's Criterion)

A sequence of reals $(x_j)_{j\geq 1}$ is u.d. mod 1 if and only if for every complex-valued 1-periodic continuous function f,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(x_j) = \int_0^1 f(x)dx.$$

That is, if and only if, for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi\, i\, t\, x_j} = 0.$$

Normal numbers and Weyl's criterion

Thus, a real x is normal to base b if and only if for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi\, i\, t\, b^j x} = 0.$$

Lemma (application of Davenport, Erdős and LeVeque's Theorem)

Let μ be a measure whose Fourier transform decays quickly. Let I be an interval and b a base. If for every non-zero integer t,

$$\sum_{n\geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i t b^j x} \right|^2 d\mu(x) \ < \infty$$

then for μ-almost all x in I,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi\, i\, t\, b^j x} = 0.$$

Low discrepancy as finitely many bounded harmonic sums

Lemma (Becher, Slaman 2013)

For any positive real ϵ there is a finite set of positive integers T and a positive real δ such that for any real x, any base b and for any n,

$$\text{if for all } t \in T,\ \left|\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i t b^j x}\right|^2 < \delta \text{ then } D(\{b^j x\} : 0 \leq j < n) < \epsilon.$$

Furthermore, such T and δ can be computed from ϵ.

Back to the main result

Theorem (Becher, Heiber, Slaman, 2014)

There is a computable absolutely normal Liouville number.

Liouville numbers

The set of *Liouville numbers* is

$$\left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall k \in \mathbb{N}\ \exists p, q \in \mathbb{N} \quad \left| x - \frac{p}{q} \right| < \frac{1}{q^k} \right\}$$

It is uncountable, with Lebesgue measure zero and Hausdorff dimension zero.
An example is Liouville's constant, $\sum_{k \geq 1} 10^{-k!}$.

In 2002 Bugeaud established the existence of absolutely normal Liouville numbers.

Kaufman-Bluhm's measure for Liouville numbers

Kaufman (1981) showed that the set of Liouville numbers supports a positive measure whose Fourier transform vanishes at infinity. Bluhm (1998,2000) gave a simpler account.

Their measure is the limit of sequence of measures μ_k, for $k \in \mathbb{N}$.

Each μ_k is supported by the set of numbers that satisfy an instance of the Liouville condition for exponent k.

Kaufman-Bluhm's measure for Liouville numbers

For every pair of integers m and k such that $k \geq 1$, let

$$E(m,k) = \bigcup_{\substack{m \leq q < 2m \\ \text{prime } q}} \left\{ x \in [0,1] : \exists p \; 0 \leq p \leq q, \; \left| x - \frac{p}{q} \right| \leq \frac{1}{q^k} \right\}$$

The Fourier transform of a real function f is $\widehat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i x t} \, dt$.

Lemma (Bluhm 1998)

There is a family of C^2 functions $g_{m,k}$, parametrized by m and k, such that

1. *support*$(g_{m,k}) \subseteq E(m,k)$,
2. $\widehat{g_{m,k}}(0) = 1$,
3. *for every function Ψ in C^2 of compact support, for every $k > 0$ and $\delta > 0$ there is M such that for every $m \geq M$ and for every $x \in \mathbb{R}$,*

$$|\widehat{(\Psi g_{m,k})}(x) - \widehat{\Psi}(x)| \;\leq\; \delta \, (1+|x|)^{-1/(2+k)} \log(e+|x|) \; \log\log(e+|x|).$$

Our Bluhm-style measures

We define Bluhm-style measures $\nu_{I,m,k}$.

For I a subinterval of $[0,1]$ we define a function F_I, a smooth version of the characteristic function of I.

- $g_{m,k}$ is the sum of similarly smoothened intervals in $E(m,k)$.
- $\nu_{I,m,k} = \int F_I g_{m,k}$.

Observe that $support(\nu_{I,m,k}) \subseteq support(g_{m,k}) \cap I \subseteq E(m,k)$.

How deep into the fractal

Recall $\langle a; b\rangle$ denotes $\lceil a/\log_2 b\rceil$.

Lemma

Let I be a dyadic interval with length 2^{-a} and let k be a positive integer. Let b be a base and let t be a positive integer. Then, there is an integer M such that for every $m \geq M$ and every positive integer ℓ,

$$\int \left| \frac{1}{\ell} \sum_{j=\langle a;b\rangle+1}^{\langle a;b\rangle+\ell} e^{2\pi i t b^j x} \right|^2 d\nu_{I,m,k}(x) \; < \; \frac{100}{\ell}.$$

Moreover, M is uniformly computable from I, k, b and t.

The proof uses Bluhm's Lemma of quick decay.

At least half of the measure goes to low discrepancy

Adapting Davenport, Erdős and LeVeque's Theorem for $\nu_{I,m,k}$:

Lemma

Let B be a set of bases, I be a dyadic interval of length 2^a and k a positive integer. Let ϵ be a positive real number. Then, there are positive integers M and L such that for every $\geq M$,

$$\nu_{I,m,k}\left(\{y \in I : \forall b \in B, \forall \ell \geq L, D(\{b^j y\} : \langle a; b\rangle \leq j < \langle a+\ell; b\rangle) < \epsilon\}\right) \geq \tfrac{1}{2}.$$

Furthermore, L is uniformly computable from k, B and ϵ, and does not depend on I; M is uniformly computable from I, k, B and ϵ.

A computable absolutely normal Liouville number

Consider a computable $(I)_{n\geq 1}$ of dyadic intervals $I_n = [x_n, x_n + 2^{-a_n})$ such that

- $I_1 = [0,1)$ and for all $n \geq 1$, $I_{n+1} \subset I_n$,
- $\lim_{n\to\infty} |I_n| = 0$,
- I_n meets the Liouville condition for exponent n.
- for every base b, $\lim_{n\to\infty} D(\{b^j x_n\} : 0 \leq j < \langle a_n; b\rangle) = 0$.

Thus, the dyadic rational x_{n+1} is obtained at step $n+1$ by concatenating the expansion of x_n with a new block of digits.

For every base $b \leq n+1$, $D(\{b^j x_{n+1}\} : \langle a_n; b\rangle \leq j < \langle a_{n+1}; b\rangle)$ is small.

Then the real $x = \bigcap_{i\geq 1} I_i = \lim_{n\to\infty} x_n$ is well defined, it is a Liouville number, and, by continuity of discrepancy, x is absolutely normal.

The End

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