Relating and contrasting plain and prefix Kolmogorov complexity

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Outline

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1. Definitions and some results
2. Relating C and K
3. Contrasting plain and prefix deficiency

Definitions

- *Plain* complexity of a bitstring: $C_U(x|y) = \min\{|p| : U(p,y) = x\}$.
  - p and y are written on the work-tape containing symbols $0, 1, b$.
  - $|p|$ can be scanned at any stage of the computation: $C(x|C(x)) = C(x) + O(1)$.
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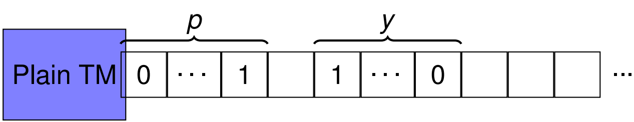

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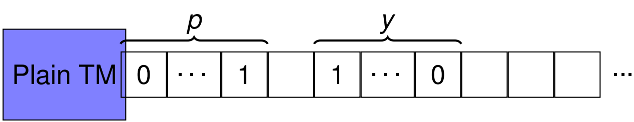

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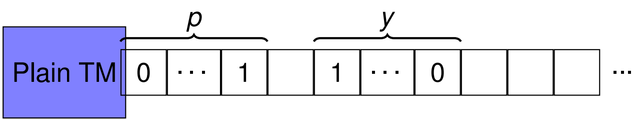

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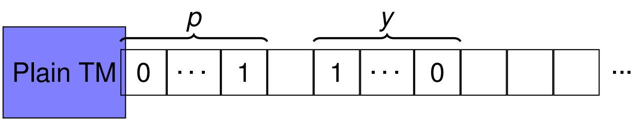

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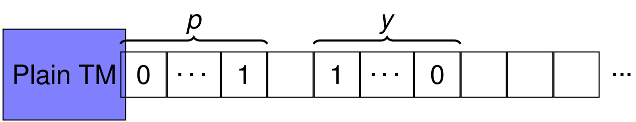

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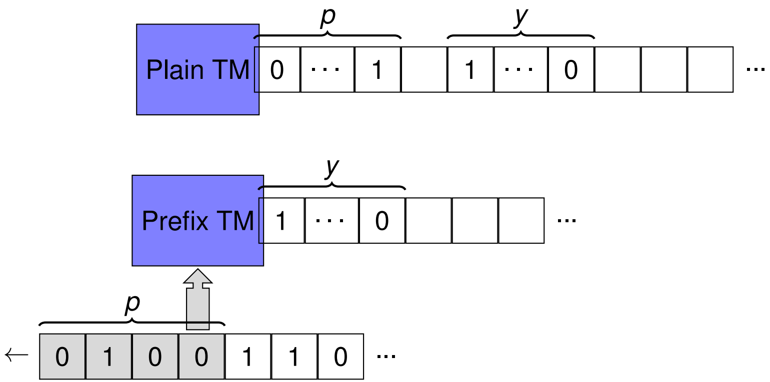

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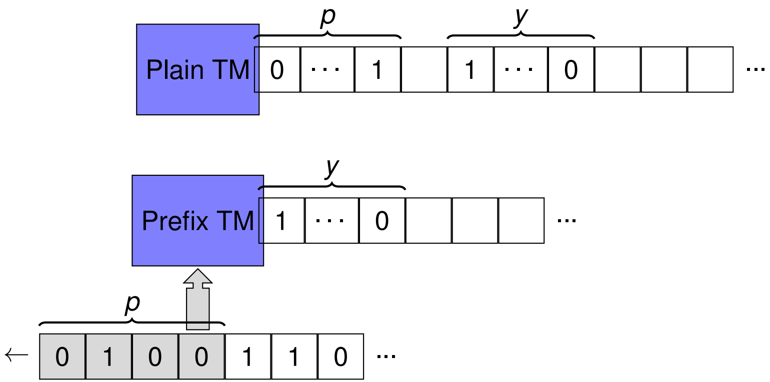

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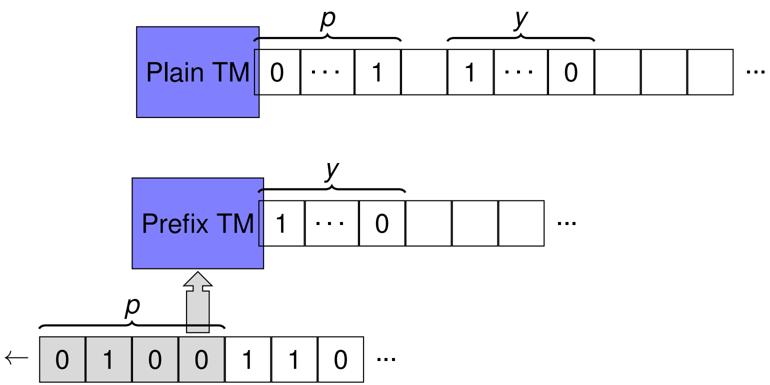

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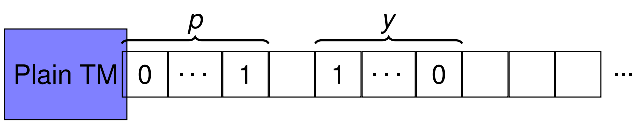

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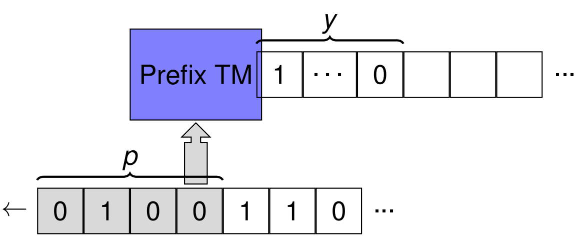

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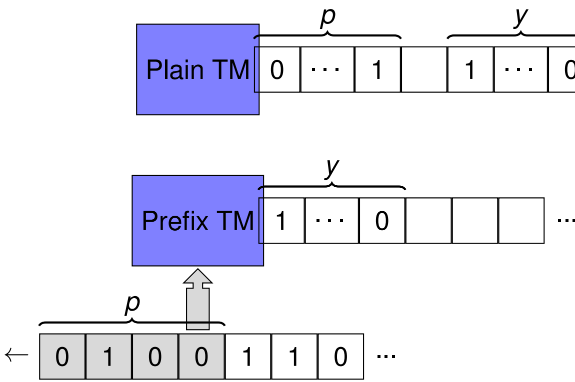

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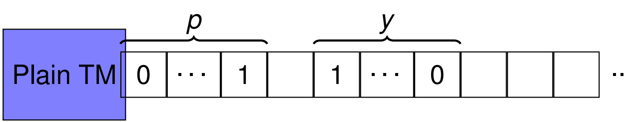

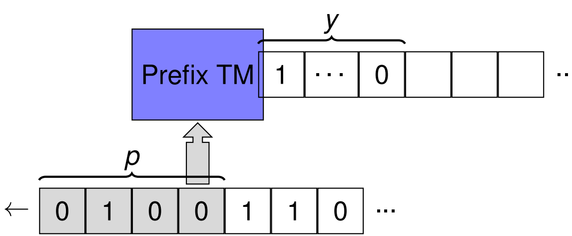

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A minimal prefix program contains more information than a minimal plain program

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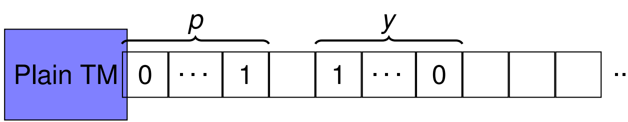

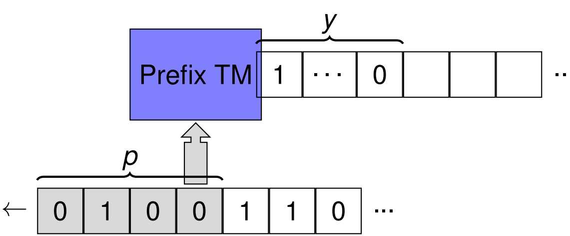

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Question: $K(C(x)) + C(x) = K(x)$ or $K(C(x)|x, K(x)) \leq O(1)$?

Minimal self-delimiting programs contain more information

Theorem (B.)

There exists a sequence ω for which $K(\omega_1 \dots \omega_N) - K(N) \leq O(1)$ for infinitely many N, and for which $C(\omega_1 \dots \omega_N) - C(N)$ tends to infinity.

(A question from [Barmpalias 2013].)

For 2-random α (i.e. Martin-Löf random relative to the halting problem), $\exists c, e$ and $\exists^\infty n$:

$$\begin{aligned} C(\alpha_1 \cdots \alpha_n) &\geq n - c & \text{i.e. } C(\cdot) \text{ is maximal} \\ K(\alpha_1 \cdots \alpha_n) &\geq n + K(n) - e & \text{i.e. } K(\cdot) \text{ is maximal} \end{aligned}$$

[Miller 2004 and 2009, Nies–Stephan–Terwijn 2004]

Nice proof using Kolmogorov complexity?
Have initial segments with maximal $C(\cdot)$ also maximal $K(\cdot)$? [Bienvenu]

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For every 3-random sequence ω there are a c and infinitely many j such that $j - C(\omega_1 \dots \omega_j) \leq c$ and $K(j) + j - K(\omega_1 \dots \omega_j) \geq \log\log j - c$.

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\begin{aligned}
K(x) &= C(x) + CC(x) + O(CCC(x)) \\
C(x) &= K(x) - KK(x) + O(KKK(x))
\end{aligned}
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We use two lemma's:

- $K(x, y) = K(x) + K(y|x, K(x))$ [Symmetry of information]
- $K(x|i) = i + c \implies C(x) = i + O(c)$

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Proof: note that $|K(x|i) - K(x|j)| \leq K(i-j) + O(1) \leq O(\log|i-j|)$.

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$$C(x)$$
$$\not{i} - K(x|i) = c + C(x) - i$$

Let $KK(x) = K(K(x))$, $CC(x) = C(C(x))$, ...

$$K(x) = C(x) + CC(x) + O(CCC(x))$$
$$C(x) = K(x) - KK(x) + O(KKK(x))$$

We use two lemma's:

- $K(x, y) = K(x) + K(y|x, K(x))$ [Symmetry of information]
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Proof: note that $|K(x|i) - K(x|j)| \leq K(i - j) + O(1) \leq O(\log|i - j|)$.

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$$\implies |C(x) - i| \leq O(c). \quad \square$$

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$$K(x) = K(K(x), x)$$

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OK

Let $KK(x) = K(K(x))$, $CC(x) = C(C(x))$, ...

$$\begin{aligned} K(x) &= C(x) + CC(x) + O(CCC(x)) \\ C(x) &= K(x) - KK(x) + O(KKK(x)) \end{aligned}$$

OK

This follows from

$$\begin{aligned} CC(x) &= KK(x) + O(KKK(x)) \\ KKK(x) &\leq O(CCC(x)). \end{aligned}$$

Let $KK(x) = K(K(x))$, $CC(x) = C(C(x))$, ...

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Apply OK with $x \longleftarrow K(x)$:

$$C(K(x)) = K(K(x)) + KK(K(x)) + O(KKK(K(x)))$$

Let $KK(x) = K(K(x))$, $CC(x) = C(C(x))$, ...

$$K(x) = C(x) + CC(x) + O(CCC(x))$$
$$C(x) = K(x) - KK(x) + O(KKK(x)) \qquad \text{OK}$$

This follows from

$$CC(x) = KK(x) + O(KKK(x))$$
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Thus
$CK(x) = KK(x) + O(KKK(x))$

Apply OK with $x \longleftarrow K(x)$:

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$$
\begin{aligned}
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Note that

$$
\begin{array}{ccccccc}
a & = & b & + & c & + & O(d) \\
 & & & \Downarrow & & & \\
C(a) & = & C(b) & + & O(K(c)) & + & O(d)
\end{array}
$$

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Note that

$$
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a &= b + c + O(d) \\
&\Downarrow \\
C(a) &= C(b) + O(K(c)) + O(d)
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K(x) &= C(x) + CC(x) + O(CCC(x)) \\
C(x) &= K(x) - KK(x) + O(KKK(x)) \qquad \text{OK}
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This follows from

$$
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CC(x) &= KK(x) + O(KKK(x)) \quad \text{OKE} \\
KKK(x) &\leq O(CCC(x)) .
\end{aligned}
$$

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$$
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CK(x) &= KK(x) + O(KKK(x)) \\
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Note that

$$\begin{array}{ccccccc} a & = & b & + & c & + & O(d) \\ & & & \Downarrow & & & \\ C(a) & = & C(b) & + & O(K(c)) & + & O(d) \\ & & & \text{i.e.} & & & \\ C(C(x)) & = & C(K(x)) & + & O(K(KK(x))) & + & O(KKK(x)) \end{array}$$

$KKK(x) \leq 2\,CKK(x)$
Because $K(y) \leq 2\,C(y)$.

Let $KK(x) = K(K(x))$, $CC(x) = C(C(x))$, ...

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i.e.

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$KKK(x) \leq 2\,CKK(x) \leq 2\,CCC(x) + O(\log KKK(x))$

Because $K(y) \leq 2\,C(y)$. Apply $C(\cdot)$ to OKE

Outline

Randomness deficiency: 2 definitions

If α is random, then $0^{1,000,000}\alpha$ is "less" random.

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- $f : \{0.1\} \to \mathbb{R}^+ \cup \{\infty\}$ is *lower semicomputable* if

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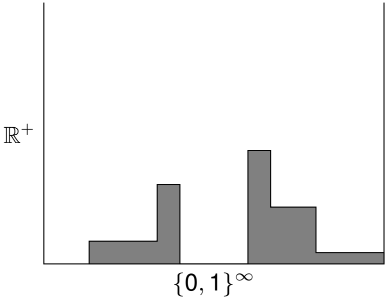

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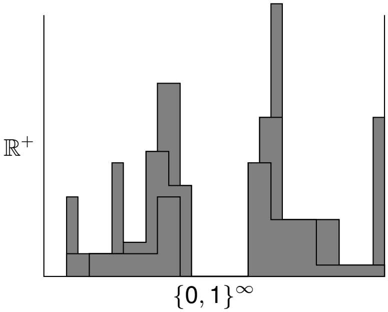

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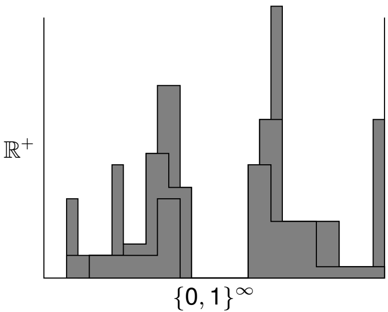

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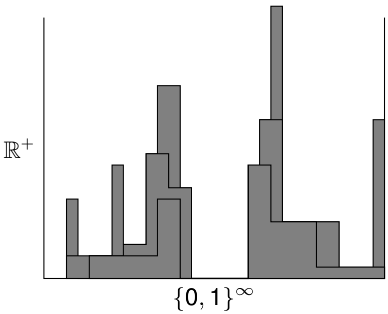

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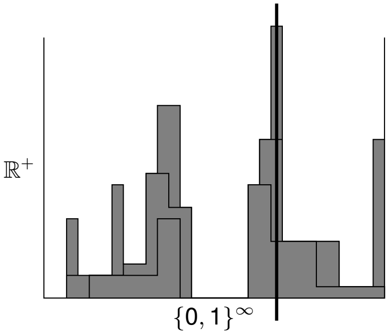

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- $f : \{0.1\} \to \mathbb{R}^+ \cup \{\infty\}$ is *lower semicomputable* if

  $$f = \sum_{i \in \mathbb{N}} f_i$$

  for a uniformly computable sequence of basic functions.
- a lower semicomputable f is an *integral test* if

  $$\int f(\alpha) d\alpha \leq O(1).$$

- an integral test f is *maximal* if for all such g: $g - f$ is bounded.

$d(\alpha) = \infty$ iff α is non-random.

Randomness deficiency: 2 definitions

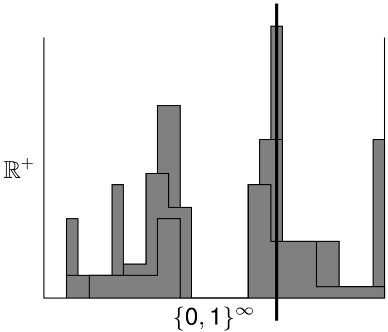
If α is random, then $0^{1,000,000}\alpha$ is "less" random.

Kolmogorov complexity:	strings	$\longrightarrow$	numbers
Randomness deficiency:	sequences	$\longrightarrow$	numbers

plain deficiency of a sequence α is

$$d_C(\alpha) = \max\{k : \alpha \in U_k\}$$

with U_k a universal Martin-Löf-test.
(the choice of U_k affects d_C by at most $O(1)$)



$d_K(\omega) = \sup_n[n - K(\omega_1 \cdots \omega_n)] + O(1)$
[Gács 1980]

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d_C and d_K vs C and K

- $d_K(\alpha, \beta) = d_K(\alpha) + d_K(\beta|\alpha, d_K(\alpha))$

  For prefix-free c.e. set $S \subset \{0,1\}^*$ one has: $d_K(x\alpha) = |x| - K(x) + d_K(\alpha|x, K(x))$.

- $K(x, y) = K(x) + K(y|x, K(x))$

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$$\begin{aligned} |d_C(\alpha_\ell) - d_C(\beta_\ell)| &\leq 0 \\ d_K(\alpha_\ell) - d_K(\beta_\ell) &\geq \ell \end{aligned}$$

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- Choose k as in Lemma 3.
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α	$d_K(\alpha)$	$d_C(\alpha)$
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Thanks for listening.