On Rogers semilattices of Analytical Hierarchy

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Numberings

Definition

A surjective mapping α of the set N of natural numbers onto a nonempty set A is called numbering of A.
$\alpha : N \to A$

Definition

The collections of all numberings of A will be denoted by $Num(A)$.

Uniform computations

Definition

A sequence $C_0, \ C_1, C_2, \dots$ of c.e. subsets of ω is called uniformly c.e. if $\{(x,i) | \ x \in C_i\}$ is c.e.

Definition

Let A is a family of c.e. sets and $\nu(0)$, $\nu(1)$, . . . is uniformly c.e. sequence then ν is called a computable numbering.

Approach of Goncharov-Sorbi (1997) - the uniformity idea

Let C be a class of constructive objects described by 'expressions' (programs) of some language L that is equipped with Godel numbering γ for 'expressions' of L.

Any partial mapping $i : L \to C$ is called interpretation for the expressions from L.

A numbering $\nu : N \to A \subseteq C$ is called computable numbering (relative to i) if there exists a computable function f such that for every $n \in \omega$, $\nu(n) = i(\gamma_{f(n)})$.

Example

$C = \Sigma^0_{n+1}$,
$L = \{\Sigma^0_{n+1} -$ formulas of arithmetics of a free variable $x\}$,
$i(\Phi) = \{a | \ N \models \Phi(\bar{a})\}$.
Then a numbering ν of a family $A \subseteq \Sigma^0_{n+1}$ is called Σ^0_{n+1}-computable if there exists a computable function f s.t., for every $m \in N$,
$\nu(m) = \{x | N \models \gamma_{f(m)}(\bar{x})\}$

Computable numberings in some hierarchies

Theorem (Goncharov and Sorbi,1997)

A numbering ν of a family S of Σ^0_{n+1} sets is Σ^0_{n+1}-computable $\Leftrightarrow$ $\{(m,x)|\ x \in \nu(m)\} \in \Sigma^0_{n+1}$.

Straightforward modifcation yields a criterion:

Definition

A numbering ν of the family of analytic subsets of the natural series is said to be Π^1_{n+1}-computable if the set $G_\nu = \{(x,y)|y \in \nu(x)\}$ is Π^1_{n+1}-set.

Definition

The set of Π^1_{n+1}-computable numberings of A will be denoted by $Com^1_{n+1}(A)$.

Reducibility

Definition

Let $\nu \in Num(A)$, $\mu \in Num(B)$, where A and B are any given families. Numbering ν is reducible to numbering μ (in symbols, $\nu \leq \mu$) if $\nu(x) = \mu f(x)$ for some computable function f and all $x \in \omega$.

Definition

If $\nu \leq \mu$ and $\mu \leq \nu$ then $A = B$ and we say that ν and μ are equivalent (in symbols, $\nu(x) \equiv \mu(x)$) numberings of A.

The equivalence class of a numbering ν is called the degree of ν, denoted by $\deg(\nu)$.

The set of all degrees of the elements of $Num(A)$ will be denoted by $L(A)$.

Rogers semilattice in analytical hierarchy

Definition

$R^1_{n+1}(A) =< Com^1_{n+1}(A)_{/\equiv}; \leq >$ is called the Rogers semilattice of A.

Number of minimal numberings in the case of the arithmetical hierarchy

Badaev and Goncharov have solved the problem of the cardinality of the set of minimal elements in $R^0_{n+2}(A)$, for any infinite family A.

Theorem (Goncharov and Badaev)

For every n, if A is an infinite Σ^0_{n+2}-computable family, then $R^0_{n+2}(A)$ has infinitely many minimal elements.

Number of minimal numberings in the case of the analytical hierarchy

Theorem

For every n there are infinitely many minimal numberings of an infinite Π^1_{n+1}-computable family S of Π^1_{n+1}-sets.

1. ν— Π^1_{n+1}-computable numbering of S.
2. $A \in S$
3. M— maximal set, $\overline{M} = \{m_0 < m_1 < ... < m_k < ...\}$.
4. Numbering

$$\nu^A_M(m) = \begin{cases} \nu(i), \text{ if } m = m_i; \\ A, \text{otherwise} \end{cases} \quad (1)$$

   is minimal.
5. $\nu^A_M \leq \nu^B_M$ iff $A = B$.

Friedberg

Theorem (Friedberg)

There exists a sequence of $S_0, S_1, S_2, \ldots$ of uniformly recursively enumerable sets in which every recursively enumerable set occurs once and only once.

Definition

If ν is a numbering of some family S, and ν is $1-1$, it is usually called Friedberg numbering.

Friedberg enumeration in analytical hierarchy, James C. Owings, JR

Theorem (Owings)

There is no meta-r.e. sequence $S(\alpha)(\alpha < \omega_1)$ *of* Π^1_1 *sets such that for each* Π^1_1 *set* A *there is one and only one* α *for which* $A = S(\alpha)$.

Friedberg enumeration in analytical hierarchy

Theorem

There is no a Π^1_{n+1}-computable Friedberg enumeration of all Π^1_{n+1}-sets.

1. If there a Friedberg enumeration of all Π^1_{n+1} sets there is a Π^1_{n+1}-computable Friedberg enumeration of all infinite Π^1_{n+1}-sets.
2. There is no a Π^1_{n+1}-computable Friedberg enumeration of all infinite Π^1_{n+1}-sets.

Corollary

There is no a Σ^1_{n+1}-computable Friedberg enumeration of all Σ^1_{n+1}-sets.

Rogers semilattices

Theorem

Elementary theory of any nontrivial Rogers semilattices of analytical hierarchy is hereditarily undecideble.

1. ϵ— the family of all c.e. sets. Partially ordered set E$\langle\epsilon, \subseteq\rangle$ is a lattice. Finite subsets of N form an ideal of that lattice. Factoring $\langle\epsilon, \subseteq\rangle$ w.r.t. this ideal yields a factor lattice, denoted by $\langle\epsilon^*, \subseteq^*\rangle$. An element of ϵ^* consisting of finite sets is denoted by 0.
2. $\hat{\mu}$ stands for the principal ideal of $R^1_{n+1}(S)$ generated by deg(μ).
3. For every numbering $\nu \in Com^1_{n+1}(S)$, there is a numbering $\mu \in Com^1_{n+1}(S)$ such that $\nu \equiv_{0'} \mu$ and
   (1) if S is finite then $\langle\hat{\mu}, \leq\rangle \cong \langle\epsilon^*, \subseteq^*\rangle$;
   (2) if S is infinite then $\langle\hat{\mu}, \leq\rangle \cong \left\langle\epsilon^* - \{0\}, \subseteq^*\right\rangle$.
4. Elementary theory of ϵ^* is hereditarily undecideble.

Rogers semilattices

Corollary

Let S be an infinite family of Σ^1_{n+1}-sets, with $Com^1_{n+1}(S) \neq \emptyset$. Then there exists a numbering $\beta \in Com^1_{n+1}(S)$ such that the principal ideal of Rogers semilattices $R^1_{n+1}(S)$ generated by $deg(\beta)$ contains no minimal elements.

Thank you for attention!