Finite automorphism bases for degree structures

Mariya I. Soskova[1]

Sofia University

joint work with Theodore Slaman

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Let $\mathcal{A}$ be a structure with domain A. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever f and g are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

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$Aut(\mathcal{D}_T)$ is countable and every member has an arithmetically definable presentation.

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Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

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   We call such a function *an indexing* of $\mathcal{Z}$.

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   A degree $\mathbf{x}$ is c.e. if it is the join of an element from $\mathcal{A}$ and an element from $\mathcal{B}$.

The goal

Theorem (Slaman and Woodin)

There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

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We will call e an index for a Δ^0_2 set X if $\{e\}^{\emptyset'}$ is the characteristic function of X.

Step 1: Reducing to low sets

Lemma

If $\mathbf{x} \leq_T 0'$ *then there are low degrees* $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4$*, such that* $\mathbf{x} = (\mathbf{g}_1 \vee \mathbf{g}_2) \wedge (\mathbf{g}_3 \vee \mathbf{g}_4)$.

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- If in $\mathcal{M}$ "e is an index of a non-low Δ^0_2 set X" then we search in $\mathcal{M}$ for indices e_1, e_2, e_3, e_4 of low Δ^0_2 sets which define the degree of X.

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- We map $e^{\mathcal{M}}$ to $(\varphi(e_1^{\mathcal{M}}) \vee \varphi(e_2^{\mathcal{M}})) \wedge (\varphi(e_3^{\mathcal{M}}) \vee \varphi(e_4^{\mathcal{M}}))$.

Step 2: Distinguishing between low Δ_2^0 sets

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There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $\mathbf{x}$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$.

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If $\mathbf{x}, \mathbf{y} \leq \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \not\leq \mathbf{x}$ then there are $\mathbf{g}_i \leq \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 degrees $\mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$ such that:

1. $\mathbf{b}_i$ *and* $\mathbf{c}_i$ *are elements of* $\mathcal{Z}$.
2. $\mathbf{g}_i$ *is the least element below* $\mathbf{a}_i$ *which joins* $\mathbf{b}_i$ *above* $\mathbf{c}_i$.
3. $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.
4. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

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1. The automorphism group of $\mathcal{D}_T(\leq \mathbf{0}')$ is countable.
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3. Every relation $\mathcal{R} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ induced by an arithmetically definable degree invariant relation is definable with finitely many Δ^0_2 parameters. If $\mathcal{R}$ is invariant under automorphisms then it is definable.

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4. $\mathcal{D}_T(\leq \mathbf{0}')$ is rigid if and only if $\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable with first order arithmetic.

Part II: The structure of the enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D \rangle \in W \ \& \ D \subseteq B)\}.$$

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- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

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A set A is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

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The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

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$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

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If $\mathbf{x} \in \mathcal{D}_T$ then we will call $\iota(\mathbf{x})$ *the image of* $\mathbf{x}$.

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The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

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- *If $\mathcal{D}_T$ is rigid then $\mathcal{D}_e$ is rigid.*
- *The automorphism analysis for the enumeration degrees follows.*
- *The total degrees below $\mathbf{0}_e^{(5)}$ are an automorphism base of $\mathcal{D}_e$.*

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2. The parameters $\vec{p}$ code an indexing of the image of a uniformly low set $\mathcal{Z}$.

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3. Every low total Δ^0_2 enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $\mathcal{Z}$.

Towards a better automorphism base of $\mathcal{D}_e$

Theorem

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total Δ^0_2 enumeration degrees.

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3. Every low total Δ^0_2 enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $\mathcal{Z}$.
4. Every total Δ^0_2 enumeration degree is uniquely positioned with respect to the low total Δ^0_2 enumeration degrees.

An improvement

Theorem

1. *Every low Δ_2^0 enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees and the low 3-c.e. enumeration degrees.*

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2. *Every low* 3*-c.e. enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees.*

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2. *Every low 3-c.e. enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees.*

If $\vec{p}$ defines a model of arithmetic $\mathcal{M}$ and an indexing of the images of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total Δ^0_2 enumeration degrees.

Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some Δ^0_2 Turing degree.

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- If we succeed then relativizing the previous step to any total Δ^0_2 enumeration degree we can extend this to an indexing of the image of $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} [\mathbf{x}, \mathbf{x}']$.

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- If we succeed then relativizing the previous step to any total Δ^0_2 enumeration degree we can extend this to an indexing of the image of $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} [\mathbf{x}, \mathbf{x}']$.
- We will use that the image of the relation ‘c.e. in’ and the enumeration jump are definable.

C.e. in and above a Δ^0_2 degree

Suppose that $\mathbf{x}$ is Δ^0_2 and $\mathbf{y}$ is c.e. in and above $\mathbf{x}$.

1. If $\mathbf{y} \geq \mathbf{0}'$ then we use Shoenfield's jump inversion theorem to find a Δ^0_2 degree $\mathbf{z}$ such that $\mathbf{z}' = \mathbf{y}$.

C.e. in and above a Δ_2^0 degree

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3. Define an indexing of all low and c.e.a. relative to $\mathbf{x}$ such avoid the cone above $\mathbf{0}'$.

- We can define the set of images of low relative to $\mathbf{x}$ degrees that are c.e. in and above $\mathbf{x}$ and avoid the cone above $\mathbf{0}'$.

C.e. in and above a Δ_2^0 degree: complicated case

Theorem

If Y and W are c.e. sets and A is a low c.e. set such that $W \not\leq_T A$ and $Y \not\leq_T A$ then there are sets U and V computable from W such that:

1. $V \leq_T Y \oplus U$
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Relative to X and with $W = \emptyset'$ we get:

Within the class of low and c.e.a degrees relative to **x** which do not compute $\emptyset'$, **y** is uniquely positioned with respect to the Δ^0_2 Turing degrees.

The rest of the total enumeration degrees

Theorem

Let $\vec{p}$ are parameters that index the image of the c.e. Turing degrees then $\vec{p}$ index $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} [\mathbf{x}, \mathbf{x}']$.

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Extend to an indexing of the image of all Δ_3^0 Turing degrees.

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Theorem

There are high Δ^0_2 degrees $\mathbf{h}_1$ and $\mathbf{h}_2$ such that every 2-generic Δ^0_3 Turing degree $\mathbf{g}$ satisfies $(\mathbf{h}_1 \vee \mathbf{g}) \wedge (\mathbf{h}_2 \vee \mathbf{g}) = \mathbf{g}$.

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Note that $\mathbf{h}_i \vee \mathbf{g} \in [\mathbf{h}_i, \mathbf{h}_i']$ thus we have a way to identify this degree and hence we have a way to identify $\mathbf{g}$.

And now we iterate!

Theorem

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} sets.

Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total Δ^0_2 enumeration degrees:

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Question

1. *Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?*

Consequences

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Question

1. *Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?*
2. *Can we extend automorphisms of the c.e. degrees to automorphisms of $\mathcal{D}_T$ or of $\mathcal{D}_e$?*