Integer-valued Randomness and Degrees

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Martingales

- Unpredictability paradigm - von Mises, 1919.
- You try to make money by betting on the next bit of the sequence. If the sequence is random, you should not be able to make arbitrarily much.
- A martingale is a function $f : 2^{<\omega} \to \mathbb{R}_{\geqslant 0}$ such that for all σ,

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$

  (fairness condition)
- A martingale f *succeeds* on A if $\limsup_n f(A \upharpoonright n) = \infty$.

- A martingale is *c.e.* if $f(\sigma)$ is left-c.e. That is, there is a computable approximation f_s where $f(\sigma) = \lim_s f_s(\sigma)$ and $f_s(\sigma)$ is an increasing sequence of rationals.

Theorem (Schnorr)

A real is Martin-Löf random iff no c.e. martingale succeeds on it.

We can vary the effectiveness of the martingale, or the definition of "succeeds" to get different randomness notions.

If $f(\sigma)$ is a computable real, this leads to computably randoms.
Computable martingale + *Schnorr succeeds* $\implies$ Schnorr random.
Computable martingale + *Kurtz succeeds* $\implies$ Kurtz random.

Integer-valued Randoms

- The martingales allow wagers of, say, $\$\frac{1}{1{,}000{,}000}$. This cannot be done in a casino.
- What if we allowed wagers that were discrete? For example, \$1, \$2, \$3, ...

Definition (Bienvenu, Stephan, Teutsch)

X is IVR iff no computable integer-valued martingale succeeds on X.

- Can also define F-valued random, finitely-valued random, and single-valued random.

Theorem (Bienvenu, Stephan, Teutsch)

1. *Computably random implies IVR implies FVR implies SVR.*
2. *Kurtz random implies SVR.*
3. *FVR implies bi-immune.*

- We know that computably random implies Schnorr implies Kurtz (and no reversals).
- And Schnorr implies law of large numbers.

Theorem (Bienvenu, Stephan, Teutsch)

No other implications hold.

- Consider the real-valued martingale which starts with \$1 and wagers half its capital on 1 every time. No matter how many times it may lose, it always has some capital left. It always then has a chance of succeeding later.
- Integer-valued martingales have a minimum bet.
- Suppose m is integer-valued and wagers some of its capital on the outcome 1. It must wager at least \$1. Then if the outcome is 0, it must lose at least \$1.
- So if m has $\$k$, it can lose at most k times before it is bankrupt and cannot wager again.
- Therefore a strategy for defeating an integer-valued martingale is finitary.

Genericity

Definition (Actually a theorem of Jockush and Posner)

A is called *n-generic* if A meets or avoids each Σ^0_n set S of strings. That is, either

- $(\exists \sigma \prec A)\sigma \in S$, or
- $(\exists \sigma \prec A)(\forall \tau \in S)(\tau \not\succ \sigma)$.

(Kurtz) B is weakly n-generic if it meets all dense S's.

Theorem (BST)

- *If A is weakly 2-generic then A is IVR. Hence the IVR sets are co-meagre.*
- *There is a 1-generic which is not IVR.*

Corollary

There is an IVR which is not Schnorr random.

Some other results

- A technique which can be used for real-valued martingales is the savings trick.
- Given a martingale m, you can define the martingale m' as follows. Every time you win \$1, you save it, and then wager with the remaining capital in the same proportion as m, until you make another dollar.

Theorem (Teutsch)

There is a set which is not IVR, but is IVR for martingales with the savings property.

- This is because we can no longer guarantee the proportions will give us integer wagers.

Theorem (Chalcraft, Dougherty, Freiling, Teutsch)

Let A and B be finite sets of computable real numbers. Then every A-valued random is B-valued random iff there is a $k \in \mathbb{Q}$ such that $B \subseteq k \cdot A$.

Peretz and Bavly investigate this for computable infinite sets.

Questions

- What degrees contain or bound IVRs?
- Do IVRs jump invert?
- Can we refine the level of genericity required? We have that weak 2- is enough, but 1- is not.
- Left-c.e. reals?
- What about partial IVRs?

Multiply generic sets

- A set is Σ^0_1 if it is the range of a partial computable function. So a set is 1-generic iff it meets or avoids the range of every partial computable function.
- Consider instead a function that is ω-c.a.
- That is, there is an order function h (computable, nondecreasing and unbounded) and a computable approximation $g(.,.)$ such that $\lim_s g(x,s) = g(x)$ and $g(x,s) \neq g(x,s+1)$ at most $h(x)$ many times.
- We say that g is monotonically h-c.a. if the approximation has $g(x,s) \preccurlyeq g(x,s+1)$.

Definition

Let h be an order. We say that A is h-multiply generic if A meets or avoids the range of every partial monotonically h-c.a. function. A is weakly h-multiply generic if it meets the range of every partial monotonically h-c.a. function with dense range.

- We look into what sets can compute multiply generics later.

Theorem

If h and h' are order functions, then if A is (weakly) h-multiply generic, it is also (weakly) h'-multiply generic. So we say A is multiply generic if it is h-multiply generic for some order h.

Theorem

If A is weakly multiply generic, then A is IVR.

The proof is a simple modification of the BST proof for weakly 2-genericity.
The converse does not hold as there are MLRs which are not weakly 1-generic.

- Something weaker will still allow us to compute an IVR.

Definition (Downey, Jockusch, Stob)

We say that a set of strings S is pb-dense if it is the range of a total function f with computable approximation $f(\sigma, s)$ such that

- $\lim_s f(\sigma, s) = f(\sigma)$
- $f(\sigma, 0) = \sigma$, and
- $|\{s : f(\sigma, s) \neq f(\sigma, s+1)\}| < p(\sigma)$ for some primitive recursive function p.

A set A is pb-generic if it meets all pb-dense sets.

Theorem

If A is pb-generic, then A is IVR.

Array noncomputable degrees

Definition (Downey, Jockusch, Stob)

A degree **a** is array noncomputable if for every function $f \leqslant_{wtt} \emptyset'$, there is a function $g \leqslant_T$ **a** such that

$$(\exists^\infty n)(g(n) > f(n)).$$

- Allows multiple permitting arguments.
- A weakening of non-low$_2$.

The c.e. ANC degrees are especially important. They are the degrees that

- Contain c.e. sets of infinitely often maximal Kolmogorov complexity. (Kummer)
- Have effective packing dimension 1. (Downey and Greenberg)
- Compute left-c.e. reals α and $B <_T \alpha$ such that if V is a presentation of α (that is, V is prefix-free, c.e., and $\alpha = \mu(V)$), then $V \leqslant_T B$. (Downey and Greenberg)
- Bound disjoint c.e. sets A and B such that every separating set for A and B computes the halting problem. (Downey, Jockusch and Stob)
- Do not have strong minimal covers. (Ishmukhametov)

Theorem (DJS)

Every ANC degree $\mathbf{a}$ *bounds a pb-generic.*

Theorem

1. *Every ANC degree* $\mathbf{a}$ *bounds an IVR.*
2. *If* $\mathbf{a}$ *is c.e. and bounds an IVR, then it is ANC.*

Degrees containing (or not containing) IVRs

So if ANC degrees bound IVRs, do all ANC degrees contain IVRs?

No.

Theorem

There is a c.e. ANC degree which does not contain an IVR.

Corollary

The IVR degrees are not closed upwards in the Turing degrees.

We know that every high degree contains a computably random, and so an IVR. Moving down one level in the high/low hierarchy, we have though

Theorem

There is a high_2 c.e. degree which does not contain an IVR.

The only c.e. degree which contains a MLR is the complete degree. We have here

Theorem

There is a low c.e. degree which contains an IVR.

In fact we have more

Theorem

For every degree c.e. in and above $\emptyset'$, there is a c.e. degree containing an IVR which jumps to it.

A closer look at multiply generics

- $\emptyset'$ computes a multiply generic.
- Every $\overline{GL_2}$ set $(A'' >_T (A \oplus \emptyset')')$ computes a multiply generic.
- To get finer results, we look at a new hierarchy defined by Downey and Greenberg.

Definition (Downey, Greenberg and Weber)

We say that a c.e. degree $\mathbf{a}$ is totally ω-c.a. if for all functions $g \leqslant_T \mathbf{a}$, g is ω-c.a. That is, there is a computable approximation $g(x, s)$ and a computable function h such that $g(x) = \lim_s g(x, s)$ and

$$|\{s : g(x, s) \neq g(x, s + 1)\}| < h(x).$$

- Every c.e. array computable degree is totally ω-c.a.
- These degrees are definable in the c.e. degrees (DGW).
- The c.e. not totally ω-c.a. degrees are exactly the degrees containing computably finitely random reals (Downey and Ng).

Theorem

1. *Every c.e. not totally ω-c.a. degree computes a multiply generic.*
2. *If a c.e. degree bounds a weakly multiply generic, then it is not totally ω-c.a.*

Outside the c.e. degrees, we seem to need something slightly stronger.

Definition

Let $h : \omega \to \omega^2$ be computable, nondecreasing and unbounded. We say that a degree $\mathbf{a}$ is uniformly totally ω^2-c.a. if for every $g \leqslant_T \mathbf{a}$ there is an h-computable approximation. That is, there is a computable approximation $g(\cdot, \cdot)$ and a uniformly computable sequence of functions $\langle o_s \rangle_{s<\omega}$ from ω to ω^2 such that

- $g(x) = \lim_s g(x, s)$,
- $o_0(x) \leqslant h(x)$,
- $o_{s+1}(x) \leqslant o_s(x)$, and
- if $g(x, s+1) \neq g(x, s)$ then $o_{s+1}(x) < o_s(x)$.

Theorem

If **a** *is not uniformly totally* ω^2*-c.a. then* **a** *computes a multiply generic.*

- These definitions can be extended to much larger computable ordinals.
- They give are a non-collapsing hierarchy of degrees within the low_2 degrees.

Left-c.e. reals

- Every high c.e. degree contains a left-c.e. computably random, and so a left-c.e. IVR.

Theorem

If X is left-c.e. and IVR, then X is of high degree.

Partial IVRs

- What if the betting strategy did not have to tell you in advance what it does? We then get partial IVRs.

Theorem

- *There is a partial IVR which is not partial computably random.*
- *There is an IVR which is not partial IVR. (In fact it can be low.)*

Theorem

Partial IVR and IVR cannot be separated in the high degrees.

Theorem

There is a Δ^0_2 IVR which does not bound a partial IVR.

Theorem

Every pb-generic is partial IVR.