Defining totality in the enumeration degrees

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9th International Conference on Computability,
Complexity and Randomness (CCR 2014)

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June 12, 2014

Enumeration reducibility

Friedberg and Rogers introduced enumeration reducibility in 1959.

Definition

We say that $A \subseteq \omega$ is enumeration reducible to $B \subseteq \omega$ ($A \leq_e B$) if from every enumeration of B we can enumerate A.

Formally: For each $f \in \omega^\omega$ with range B, there is a $g \in \omega^\omega$ such that $g \leq_T f$ and $A = \text{range}(g)$.

It is equivalent if we demand that there is a *uniform* way to produce an enumeration of A from an enumeration of B (Selman, 1971).

This leads to the standard definition:

Definition

$A \leq_e B$ iff there is a c.e. set W such that

$$A = \{n \colon (\exists e)\ \langle n, e\rangle \in W \text{ and } D_e \subseteq B\}.$$

(D_e denotes the eth finite set in a canonical enumeration.)

The enumeration degrees

Definition

$A \leq_e B$ iff there is a c.e. set W such that
$$A = \{n \colon (\exists e)\ \langle n, e\rangle \in W \text{ and } D_e \subseteq B\}.$$

The degree structure $\mathcal{D}_e$ induced by $\leq_e$ is called the enumeration degrees. It is an upper semi-lattice with a least element (the degree of all c.e. sets) and a jump operation.

How do the enumeration degrees relate to the Turing degrees?

Relation	Uses from B	Produces for A
$A \leq_e B$	positive information	positive information
A is B-c.e.	complete information	positive information
$A \leq_T B$	complete information	complete information

Proposition

$A \leq_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The total degrees

Proposition

$A \leq_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition

The embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound, and the jump operation.

Definition

The total degrees are the image of the Turing degrees under this embedding (i.e., they are the enumeration degrees that contain a set of the form $A \oplus \overline{A}$).

Question (Rogers, 1967)

Is the set of total enumeration degrees first order definable in the structure of the enumeration degrees?

Partial solutions

Question (Rogers, 1967)

Is the set of total enumeration degrees first order definable in the structure of the enumeration degrees?

- Kalimullin, 2003: The enumeration jump is first order definable in the enumeration degrees.

  Thus the total enumeration degrees above $\mathbf{0}_e'$ can be defined as the image of the enumeration jump.
- Ganchev and Soskova, 2010: The total enumeration degrees below $\mathbf{0}_e'$ are first order definable in the enumeration degrees.

Main ingredient: Kalimullin pairs.

- Soskova, 2013: The total enumeration degrees are first order definable with one parameter.

Main ingredient: An analysis of the automorphism group of the enumeration degrees (based on Slaman and Woodin's framework).

Answering the question

Question (Rogers, 1967)

Is the set of total enumeration degrees first order definable in the structure of the enumeration degrees?

Answer (Cai, Ganchev, Lempp, M., Soskova)

Yes.

In particular, we show that a nonzero enumeration degree is total iff it is the join of a maximal Kalimullin pair.

Our next goal is to understand what this means.

Semi-computable sets

Definition (Jockusch, 1966)

$A \subseteq \omega$ is semi-computable if there is a total computable selector function $s_A \colon \omega^2 \to \omega$, such that for all $n, m \in \omega$

- $s_A(n, m) \in \{n, m\}$, and
- if $\{n, m\} \cap A \neq \emptyset$, then $s_A(n, m) \in A$.

Intuition. $s_A(n, m)$ selects which element of $\{n, m\}$ is "more likely" to be in A.

- If A is a *left cut* in a computable linear ordering $\mathcal{L}$, then it is semi-computable.

  Let $s_A(n, m) = \min\{n, m\}$.

- Conversely, every semi-computable set A is a left cut in some computable linear ordering.

Semi-computable sets (cont.)

Every Turing degree contains a semi-computable set:

- Order $2^{<\omega}$ as follows: $\sigma \leq \tau$ if $\sigma \preceq \tau$ or σ is left of τ. (You may recognize this as the usual priority ordering.)
- Similarly, $\sigma \leq A$ if $\sigma \leq \tau$ for some $\tau \prec A$.
- For a set A, let $L_A = \{\sigma \in 2^{<\omega} : \sigma \leq A\}$.
- Then L_A is semi-computable and $L_A \equiv_T A$.

Slightly stronger:

Theorem (Jockusch, 1966)

Every nonzero Turing degree contains a semi-computable set that is neither c.e. nor co-c.e.

Theorem (Arslanov, Cooper, Kalimullin, 2003)

If A is a semi-computable set, then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

Kalimullin pairs

Theorem (Arslanov, Cooper, Kalimullin, 2003)

If A is a semi-computable set, then for every X:

$$(d_e(A) \vee d_e(X)) \wedge (d_e(\overline{A}) \vee d_e(X)) = d_e(X).$$

So $d_e(A)$ and $d_e(\overline{A})$ form a minimal pair in a very strong sense.

Kalimullin *characterized* this strong minimal pair property.

Definition (Kalimullin, 2003)

A pair of sets $\{A, B\}$ is called a $\mathcal{K}$-pair if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

In other words:

- If $\langle n, k \rangle \in W$, then $n \in A$ or $k \in B$.
- If $\langle n, k \rangle \notin W$, then $n \notin A$ or $k \notin B$.

Kalimullin pairs (cont.)

Definition (Kalimullin, 2003)

A pair of sets $\{A, B\}$ is called a $\mathcal{K}$-pair if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Examples

- If B is c.e. and A is any set, then $\{A, B\}$ is a $\mathcal{K}$-pair witnessed by $W = \mathbb{N} \times B$. Call a $\mathcal{K}$-pair with a c.e. member trivial.
- If A is a semi-computable set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair witnessed by $W = \{\langle n, k\rangle \colon s_A(n, k) = n\}$.

Proposition

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair, then $A \leq_e \overline{B}$ (and $B \leq_e \overline{A}$).

Proof.

We claim that $A = \hat{A} := \{n \colon (\exists k \in \overline{B})\ \langle n, k\rangle \in W\}$. It is clear that $\hat{A} \subseteq A$. Suppose that $n \in A \setminus \hat{A}$. Then $k \in B$ iff $\langle n, k\rangle \in W$. But this means that B is c.e., which is a contradiction. $\square$

Properties of $\mathcal{K}$-pairs

As promised:

Theorem (Kalimullin, 2003)

A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) := (\forall \mathbf{x} \in \mathcal{D}_e)[\ (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}\ ].$$

Corollary

Fix A. The set of all B, such that $\{A, B\}$ forms a $\mathcal{K}$-pair is an ideal, i.e., it is closed under join and closed downward with respect to enumeration reducibility.

Kalimullin introduced $\mathcal{K}$-pairs to give a first order definition of the jump in $\mathcal{D}_e$.

Theorem (Kalimullin, 2003)

$\mathbf{0}_e'$ is the largest degree that is the join of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{a}, \mathbf{c})$.

Maximal $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

- Maximal $\mathcal{K}$-pairs are first order definable in $\mathcal{D}_e$.
- If $\{A, \overline{A}\}$ is a nontrivial $\mathcal{K}$-pair, then it is maximal.
  Proof. Suppose $\{C, D\}$ is a $\mathcal{K}$-pair with $A \leq_e C$ and $\overline{A} \leq_e D$. Then by the ideal property, $\{A, D\}$ is a nontrivial $\mathcal{K}$-pair. But then $D \leq_e \overline{A}$, so in fact, $\overline{A} \equiv_e D$. Similarly, $A \equiv_e C$. □
- If A is a semi-computable set that is neither c.e. nor co-c.e., then $\{A, \overline{A}\}$ form a maximal $\mathcal{K}$-pair.
- Every nonzero total degree is the join of a maximal $\mathcal{K}$-pair.
  Proof. Consider a total degree $\mathbf{d}$. Jockusch proved that there is a semi-computable set A that is neither c.e. nor co-c.e. such that $\mathbf{d} = \deg_e(A \oplus \overline{A}) = \iota(\deg_T(A))$. Then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair and $\mathbf{d} = \deg_e(A) \vee \deg_e(\overline{A})$. □

Defining the total degrees (local version)

- If $\{A, \overline{A}\}$ is a nontrivial $\mathcal{K}$-pair, then it is maximal.
- Every nonzero total degree is the join of a maximal $\mathcal{K}$-pair.

Theorem (Ganchev and Soskova)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e(\leq \mathbf{0}'_e)$, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Therefore, if $\{\mathbf{a}, \mathbf{b}\}$ is a maximal $\mathcal{K}$-pair in $\mathcal{D}_e(\leq \mathbf{0}'_e)$, then there is a (semi-computable) set C such that $C \in \mathbf{a}$ and $\overline{C} \in \mathbf{b}$.

Corollary

A nonzero degree in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is total if and only if it is the join of a maximal $\mathcal{K}$-pair.

Theorem (Ganchev and Soskova, 2012)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Corollary

The total degrees below $\mathbf{0}'_e$ are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Defining the total degrees (global version)

- If $\{A, \overline{A}\}$ is a nontrivial $\mathcal{K}$-pair, then it is maximal.
- Every nonzero total degree is the join of a maximal $\mathcal{K}$-pair.

Theorem (Cai, Ganchev, Lempp, M., Soskova)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Therefore, if $\{\mathbf{a}, \mathbf{b}\}$ is a maximal $\mathcal{K}$-pair, then there is a (semi-computable) set C such that $C \in \mathbf{a}$ and $\overline{C} \in \mathbf{b}$.

Corollary

A nonzero degree is total if and only if it is the join of a maximal $\mathcal{K}$-pair.

Theorem (Kalimullin, 2003)

The class of $\mathcal{K}$-pairs below is first order definable in $\mathcal{D}_e$.

Corollary

The total degrees are first order definable in $\mathcal{D}_e$.

Main theorem

Theorem (Cai, Ganchev, Lempp, M., Soskova)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof.

- C will be a left cut in the computable linear ordering $(\mathbb{Q}, \leq)$.
- Let W witness the fact that $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair.
- We take two copies of the natural numbers: $\mathbb{N}$ for A and $\mathbb{N}$ for B.
- Using W, we dynamically label elements of $\mathbb{Q}$ with the elements of $\mathbb{N} \cup \mathbb{N}$.
- A rational can have at most one label, but many rationals will be given the same label.

The labeling and the cut

- C will be a left cut in the computable linear ordering $(\mathbb{Q}, \leq)$.
- Using W, we dynamically label elements of $\mathbb{Q}$ with the elements of $\mathbb{N} \cup \mathbb{N}$.

Note. The labeling will depend only on W! The choice of C will depend on $\{A, B\}$.

- Goal: produce a labeling and a cut C such that

$$A = \{m \colon (\exists q \in C)[\ q \text{ is labeled by } m\ ]\},$$
$$B = \{k \colon (\exists q \in \overline{C})[\ q \text{ is labeled by } k\ ]\}.$$

- In this case:

$$A \leq_e C$$
$$B \leq_e \overline{C}$$

The dead zone lemma

Note that our cut C cannot split a permanent dead zone. Our main lemma says that this is okay.

Say that two current labels are connected if they are in the same connected union of permanent dead zones.

Dead Zone Lemma

- If the current label n is connected to the current label k then $n \in A \iff k \notin B$.
- If the current label n is connected to the current label m then $n \in A \iff m \in A$.
- If the current label k is connected to the current label j then $k \notin B \iff j \notin B$.

In other words, for any collection of connected current labels, either all A-labels correspond to numbers in A and all B-labels correspond to numbers *not* in B, or visa versa.

Finally, we define C

Definition

The cut C is the set of all rationals q such that there is a $k \in \overline{B}$:

- q is to the left of a k-labelled rational, or
- q is in the same permanent deadzone with a k-labelled rational.

We can show that everything works out. In other words:

$$A = \{m \colon (\exists q \in C)[\ q \text{ is labeled by } m\ ]\},$$
$$B = \{k \colon (\exists q \in \overline{C})[\ q \text{ is labeled by } k\ ]\}.$$

This completes the proof. $\square$

Defining "c.e. in"

Definition

A Turing degree $\mathbf{a}$ is *c.e. in* a Turing degree $\mathbf{x}$ if there is an $A \in \mathbf{a}$ that is $\mathbf{x}$-c.e.

Theorem (Ganchev and Soskova)

Let $\mathbf{a}$ and $\mathbf{x}$ be Turing degrees such that $\mathbf{a}$ is not c.e. Then $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if there is a nontrivial $\mathcal{K}$-pair $\{C, \overline{C}\}$ such that $d_e(C) \leq_e \iota(\mathbf{x})$ and $\iota(\mathbf{a}) = d_e(C) \vee d_e(\overline{C})$.

What about when $\mathbf{a}$ is c.e.?

Theorem (Cai, Ganchev, Lempp, M., Soskova)

The set $\mathcal{CE} = \{\iota(\mathbf{a})\colon \mathbf{a} \in \mathcal{D}_T \text{ is c.e.}\}$ is first order definable in $\mathcal{D}_e$. In particular,

$$\mathbf{a} \text{ is c.e. iff } (\forall \mathbf{b} \not\leq \mathbf{0}')[\, \mathbf{a} \vee \mathbf{b} \text{ is c.e. in } \mathbf{b}\,].$$

Corollary

The image of the relation "c.e. in" in the enumeration degrees is first order definable.

Applications to automorphisms

- The total degrees are fixed by any automorphism of the enumeration degrees.
- An enumeration degree is determined by the total degrees above it (Selman, 1971).
- Therefore, the total degrees are a (definable) automorphism basis for the enumeration degrees.
- Slaman and Woodin proved that there are only countably many automorphisms of $\mathcal{D}_T$, hence the same holds for $\mathcal{D}_e$. (This was already proved by Soskova using Slaman and Woodin's framework.)
- Slaman and Woodin proved that every automorphism of $\mathcal{D}_T$ fixes the cone above $\mathbf{0}''$, hence the same holds for the enumeration degrees. (This was previously proved for the cone above $\mathbf{0}_e^{(4)}$ by Ganchev and Soskov, 2009.)
- Soskova and Slaman recently used the definability of the total degrees and "c.e. in" to show that $\mathcal{D}_e$ has a finite automorphism basis of Δ_2^0 total degrees.

— THANK YOU —