Reverse Mathematics On Quasi-Riesz Spaces

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Outline

Abstarct

- In the first part, we study the Reverse Mathematics On Quasi-Riesz Spaces. We show some trival facts.
- In the second part, we study the Reverse Mathematics On Measure theory, we prove that the first Borel-Cantelli lemma is equivalent to WWKL over RCA_0. We also give some applications.

I. Definitions

Definition (Directed sets)

Let $(A, \leq)$ be a partially ordered set. A set $B \subset A$ is directed upwards, $B \uparrow$ if for every pair a, b of elements of B there is a $c \in B$ such that $a \leq c$ and $b \leq c$.

$B \uparrow a$ to mean that $B \uparrow$ and that $\sup B = a$. Observe, for instance, that $\{a\} \uparrow a$ for every $a \in A$.

Definition (Order-closed sets)

If $(A, \leq)$ is a partially ordered set and $B \subset A$, we shall write

$$\mathcal{F}B = \{a : \ \exists\ C \subseteq B, (C \neq \emptyset \& C \uparrow a \ in\ A)\}$$

$$\mathcal{D}B = \{a : \ \exists\ C \subseteq B, (C \neq \emptyset \& C \downarrow a \ in\ A)\}$$

Then $B \subseteq \mathcal{F}B$ and $B \subseteq \mathcal{D}B$.

B is order-closed if $\mathcal{F}B = B = \mathcal{D}B$.

Lemma (ACA_0)

The following are equivalent to each other.

- $\Pi^1_1 - CA_0$.
- *Any subset B of a poset has $\mathcal{F}B$.*
- *Any subset B of a poset has $\mathcal{D}B$.*

Proof.

- (1) $\rightarrow$ (2) is trivial.
- (2) $\rightarrow$ (1). Let a Σ^1_1-formula $\psi(n)$ be of the form $\exists f \forall m \psi_0(n, f[m])$ where $\psi_0 \in \Sigma^0_0$. Let $A = \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}$. Define $\leq$ on A by the following: for $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$,
  $\sigma \leq n$ iff $\psi_0(n, \sigma^*)$ where $\sigma =< n >^\frown \sigma^*$ for some $\sigma^* \in \mathbb{N}^{<\mathbb{N}}$.
  $\sigma \leq \tau$ iff τ is extends of σ.
  Let $B = \mathbb{N}^{<\mathbb{N}}$. Then, $n \in \mathcal{F}B \leftrightarrow \psi(n)$. So, by $\Delta^0_1 - CA$, $\{n : \psi(n)\}$ exists.

□

Definition

A partially ordered linear space is a quadruple $(E, +, \cdot, \leq)$ where $(E, +, \cdot)$ is a linear space over the field $\mathbb{Q}$ and $\leq$ is a partial ording on E such that

(i) if $x \leq y$, then $x + z \leq y + z$ for every $z \in E$;

(ii) If $x \geq 0$ in E, then $ax \geq 0$ whenever $\alpha \geq 0$ in $\mathbb{Q}$.

Usually, a partially ordered linear space is defined as a linear space over $\mathbb{R}$. We will introduce such a standard case in the near future.

Lemma (RCA_0)

Let E be a partially ordered linear space, $x \in E$, A, $B \subseteq E$. Then

(i) $\sup(x + A) = x + \sup A$ *if either side exists.*

(ii) $\sup(-A) = -\inf A$ *if either side exists.*

(iii) $\sup(A + B) = \sup A + \sup B$ *if the right-hand side exists.*

(iii) *If* $\alpha \geq 0$, $\sup(\alpha A) = \alpha \sup A$ *if the right-hand side exists.*

Definition

A quasi-Riesz space is a partially ordered linear sapce $(E, +, \cdot, \leq)$ such that $(E, \leq)$ is a lattice.
If E is a quasi-Riesz sapce, we write

$$x^+ = x \vee 0, x^- = (-x) \vee 0, |x| = x \vee (-x)$$

for any $x \in E$.

The basic properties are proved in RCA_0, for example:

Let E be a quasi-Riesz space, x, y and z members of E, and α and β numbers.

- $(x \wedge y) + z = (x + z) \wedge (y + z)$.
- If $\alpha \geq 0, \alpha x \wedge \alpha y = \alpha(x \wedge y)$.
- $x + y = x \vee y + x \wedge y$.
- $x = x^+ - x^-$.
- $|x| = x^+ + x^-$.
- etc.

Definition (Solid)

Let E be a quasi-Riesz space. A set $A \subset E$ is solid if $y \in A$ whenever there is an $x \in A$ such that $|y| \leq |x|$.

If A is any subset of E, the set $\{y : \exists x \in A, |y| \leq |x|\}$ is solid; it is the smallest solid set including A, and is called the solid hull of A.

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Lemma (ACA_0)

Any subset A of a quasi-Riesz space E has a solid hull.

Question

Does this lemma imply ACA_0 over RCA_0?

Lemma (RCA_0)

Let E be a Quasi-Riesz space.

- *If x, y and z belong to E^+,*

$$z \wedge (x + y) \leq z \wedge x + z \wedge y.$$

- *If $\langle x_i \rangle_{i<n}$ is a finite sequence in E^+ and $|y| \leq \sum_{i<n} x_i$, then there is a finite sequence $\langle y_i \rangle_{i<n}$ in E such that $y = \sum_{i<n} y_i$ and $|y_i| \leq x_i$ for every $i < n$.*
- *If $\langle x_i \rangle_{i<n}$ and $\langle y_j \rangle_{j<m}$ are finite sequences in E^+ such that $\sum_{i<n} x_i = \sum_{j<m} y_j$, then there is a double sequence $\langle Z_{ij} \rangle_{i<n,j<m}$ in E^+ such that $x_i = \sum_{k<m} z_{ik}$ and $\sum_{k<n} z_{kj}$ for every $i < n$ and $j < m$.*

Proposition

The convex hull of a solid set, if exists, is solid.

V. Reverse Mathematics on Measure Theory

The measure

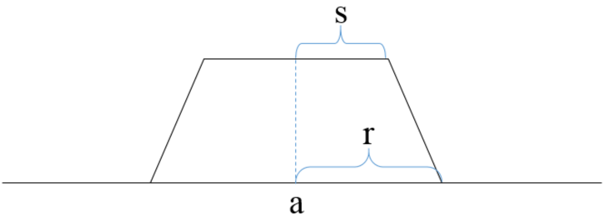
A complete separable metric space $\hat{A} = (A, d)$ is coded by the countable dense set $A \subset \mathbb{N}$ and the pseudo-metric d on A. A point of $\hat{A}$ is a strong Cauchy sequence $\langle a_n : n \in \mathbb{N} \rangle$ in the sense that $d(a_n, a_m) \leq 2^{-n}$ for any $n \leq m$.

A complete separable metric space $\hat{A}$ is compact if there exists an infinite sequence of finite sequences of $\hat{A}$, $\langle \langle x_{ij} : i \leq n_j \rangle : j \in \mathbb{N} \rangle$, such that for all $z \in \hat{A}$ and $j \in \mathbb{N}$ there exists $i \leq n_j$ such that $d(x_{ij}, z) < 2^{-j}$.

A basic open ball for this space $\hat{A}$ is coded by $\langle a, r \rangle$, where $a \in A$ and $r > 0$ is a rational. Open and closed subsets are coded by sequences of basic open balls.

A basic function on $\hat{A}$ is a code $p = \langle a, r, s \rangle$ where $a \in A$ and $r, s \in \mathbb{Q}$ which satisfy that $0 \leq s < r$. It is understood that $p = \langle a, r, s \rangle$ can be seen as a continuous function such that for any $x \in \hat{A}$,

$$p(x) = \begin{cases} 1 & \text{if } d(a,x) \leq s, \\ (r - d(a,x))/(r - s) & \text{if } s < d(a,x) < r, \\ 0 & \text{if } d(a,x) \geq r. \end{cases}$$



$C(\hat{A})$: the space of continuous real-valued functions

Let P be the set of linear combinations of basic functions with rational coefficients. Then P forms a vector space over the rational field.

The space of continuous real-valued functions $C(\hat{A})$ is defined as the complete separable Banach space $\hat{P} = (P, ||\cdot||_\infty)$. If $f = \langle p_n : n \in \mathbb{N} \rangle \in C(\hat{A})$, it is understood that $f(x) = \lim_n p_n(x)$ for any $x \in \hat{A}$.

Measure

A (probability) measure is a (code for a) positive linear functional μ on $C(\hat{A})$ such that $\mu(1) = 1$. For any open subset U of $\hat{A}$, the measure of U is defined to be

$$\mu(U) = \sup\{\mu(g) : g \prec U\},$$

where $g \prec U$ is used for the statement that $0 \leq g \leq 1$ and $g(x) = 0$ for any $x \notin U$. Similarly, for any closed subset C of X,

$$\mu(C) = \inf\{\mu(g) : C \prec g\}$$

V. The Borel-Cantelli lemma

The Borel-Cantelli lemma is a theorem about sequences of events, named after Emile Borel and Francesco Paolo Cantelli, who found it in the first decades of the 20th century.

Theorem

The following statement, called the first Borel-Cantelli lemma is equivalent to WWKL *over* RCA_0*: Let* O_n *be the sequences of open set. If* $\sum_{n=0} \mu(O_n) < \infty$*, then*

$$\mu(\bigcap_n \bigcup_{k>n} O_k) = 0.$$

Proof.

(i) Let $\sum_{n=0} \mu(O_n) < \infty$. For any $\epsilon > 0$, There exists an n_0 such that $\sum_{k=n_0}^{\infty} \mu(O_k) < \epsilon$. By WWKL, $\mu(\cup_{k=n_0}^{\infty} O_k) \leq \sum_{k=n_0}^{\infty} \mu(O_k) < \epsilon$.

(ii) We use the fact that the following statement implies WWKL: $\sum_{i=0} \mu(a_i, b_i) < 1$ implies $\langle (a_i, b_i) : i \in \mathbb{N} \rangle$ does not cover $[0, 1]$. Assume that $[0, 1] \subset \bigcup (a_i, b_i)$ and $\sum_{i=1}^{\infty} \mu((a_i, b_i)) < 1 - \epsilon$. By the Borel-Cantelli lemma, there is n such that $\mu(\cup_{i>n}(a_i, b_i)) < \epsilon$. So

$$1 \leq \mu([0, 1]) \leq \mu(\cup_{i \leq n}(a_i, b_i)) + \mu(\cup_{i>n}(a_i, b_i))$$

$$\leq \sum_{1 \leq n} \mu((a_i, b_i)) + \mu(\cup_{i>n}(a_i, b_i)) < 1,$$

which is a contradiction.

□

A related result, sometimes called the second Borel-Cantelli lemma, is a partial converse of the first Borel-Cantelli lemma.

Theorem (RCA_0)

Let O_n be the independent sequences of open sets. If $\sum_{n=0} \mu(O_n) = \infty$, then

$$\mu(\bigcap_n \bigcup_{k>n} O_k) = 1.$$

Theorem

The following assertions are pairwise equivalent over RCA_0.

(i) ACA_0.

(ii) *Let* $\langle f_n : n \in \mathbb{N} \rangle$ *be a dominated Cauchy sequence of* $L_1(\hat{A}, \mu)$ *in probability. Then, there exists subsequences which converge to some* $f \in L_1(\hat{A}, \mu)$ *a.e.*

Proof

$(i \to ii)$ Give a sequence $\langle f_n : n \in \mathbb{N} \rangle$, where $f_n = \langle P_{n,l} : l \in \mathbb{N} \rangle \in L_1(\hat{A}, \mu)$. Assume that this sequence is dominated and cauchy in probability, that is,

$$\forall \varepsilon > 0, \lim_{n,m \to \infty} \mu(\{x \in \hat{A} : |f_n(x) - f_m(x)| > \varepsilon\}) = 0.$$

By ACA, we let $g(n)$ be the least m such that $\forall l \geq m$,

$$\mu(\{x \in \hat{A} : |P_{l,n+2}(x) - P_{m,n+2}(x)| > 2^{-n-2}\}) < 2^{-n}.$$

Then define $h : \mathbb{N} \to \mathbb{N}$ by $h(0) = g(0)$ and $h(n+1) = g(m)$ where m is the least number s.t $g(m) > h(n)$.

Continue of Proof

Let $E_k = \{x \in \Omega : |P_{h(k+1),k+2}(x) - P_{h(k),k+2}(x)| > 2^{-k-2}\}$. Since $\mu(E_k) < 2^{-k}$, by the first Borel-Cantelli lemma, $\mu(\bigcap_n \bigcup_{k \geq n} E_k) = 0$. So, $\langle f_{h(k)} : k \in \mathbb{N} \rangle$ is point-wise convergent a.e.
By Lebesgue dominated convergence theorem, $\exists f \in L_1(\hat{A}, \mu)$ such that $f_{h(k)} \to f$ a.e., that is, in $\| \ \|_1$.

$(ii \to i)$ As the proof of Theorem III 2.2 in Simpson's book.

Lemma ($WWKL_0$)

Let $\langle f_n : n \in \mathbb{N} \rangle$ be a sequence of continuous functions converging to a continous function f in probability, then $\langle f_n : n \in \mathbb{N} \rangle$ point-wise converges to f a.e.

Theorem (WWKL)

Let $\langle X_n : n \in \mathbb{N}\rangle$ be an independent sequence of random variables with the same expected value m. If there exists $M > 0$ such that $\forall n(E(|X_n - \mu|^4) < M)$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} X_i(\omega) = m \text{ a.e.}$$

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Question

Does the strong law of large numbers imply WWKL_0*?*

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Thank you very much!