Themes:

- (wild) non-abelian Hodge correspondence on curves & hyperkähler moduli spaces
- example moduli spaces on $\mathbb{P}^1$ $\left(\mathcal{M}^*_{DR} \subset \mathcal{M}_{DR}\right)$
- symplectic geometry of wild character varieties
- nonlinear symplectic braid group actions

(• Logahoric connections & Grothendieck-Brieskorn-Springer)

Wild nonabelian Hodge theory on curves

Choose
- $G = GL_n(\mathbb{C})$, $T \subset G$

"irregular curve" or "wild Riemann surface"
- Σ compact smooth complex algebraic curve
- $a_1, \ldots, a_m \in \Sigma$ distinct points
- irregular types Q_i at a_i, $i = 1, \ldots, m$

Definition If $a \in \Sigma$, an irregular type Q at a is an element $Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\mathcal{O}})$

If z is a local coordinate vanishing at a

$$\hat{\mathcal{O}} = \mathbb{C}[\![z]\!], \quad \hat{K} = \mathbb{C}(\!(z)\!)$$

$$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t} = \mathrm{Lie}(T)$$

Wild nonabelian Hodge theory on curves

Choose

- $G = GL_n(\mathbb{C})$, $T \subset G$
- $\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ irregular curve

Wild nonabelian Hodge theory on curves

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- $\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ irregular curve
- weights $\theta_1, \dots, \theta_m \in \mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R} \subset \mathfrak{t}$

$$\left((\theta_i)_{jj} \in [0,1) \quad j = 1, \dots, n \right)$$

Let $\mathfrak{h}_i = C_{\mathfrak{g}}(Q_i) \subset \mathfrak{g}$ (centraliser)

- adjoint orbits $\mathcal{O}_i \subset \mathfrak{l}_i := C_{\mathfrak{h}_i}(\theta_i) = C_{\mathfrak{g}}(Q_i, \theta_i)$

Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $\mathfrak{p}_\theta \subset \mathfrak{g}$

$$\mathfrak{p}_\theta(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \;\middle|\; \lim_{z \to 0} z^{\theta} X z^{-\theta} \text{ along any ray exists} \right\}$$

Wild nonabelian Hodge theory on curves

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$$\mathfrak{p}_\theta(\mathfrak{g}) = \mathrm{Stab}(\mathcal{F}_\theta), \quad (\mathcal{F}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \left(\begin{matrix}\text{eigenspaces} \\ \text{of } \theta\end{matrix}\right)$$

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& similarly $\mathfrak{P}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & $\mathfrak{l}_i$ is Levi of $\mathfrak{P}_{\theta_i}(\mathfrak{h}_i)$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \to \Sigma$ rank n holom. vector bundle
- $\nabla : V \to V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \dots, V_{a_m}$

such that:

Near a_i V has a local trivialization in which

- $\nabla = d - A$, $A = dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$ for some $\Lambda_i \in \mathfrak{h}_i$
- $\mathcal{F}_i \cong$ standard flag $\mathcal{F}_{\theta_i}$
- Λ_i preserves $\mathcal{F}_i$ (i.e. $\Lambda_i \in \mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$)
- $\pi(\Lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ $(\pi : \mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \twoheadrightarrow \mathfrak{l}_i)$

Thm (Biquard-B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Sabbah, ...)

The moduli space $\mathcal{M}_{DR}(\Sigma, \underset{\sim}{\theta}, \underset{\sim}{O})$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
- complete if $\underset{\sim}{\theta}, \underset{\sim}{O}$ sufficiently generic

- Higgs fields should look like $-\frac{1}{2}dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$ near a_i
- same 'rotation' of the weights/eigenvalues as in Simpson 1990

Simpson's table (JAMS '90) (notation & extension to other G / parahoric case, PB '10)

	Dolbeault / Higgs	DR / Connections	Betti / monod.	
weights $t_{\mathbb{R}}$	$-\tau$	θ	$\phi = \theta + \tau$	
eigenvalues $t_{\mathbb{C}}$	$-\frac{1}{2}(\phi + \sigma)$	$\tau + \sigma$	$\exp\left(2\pi i(\tau+\sigma)\right)$	

(eigenvalues of Γ)

$t_{\mathbb{R}}$ $\quad$ $it_{\mathbb{R}}$

(eigenvalues of Λ)

$$\mathrm{Pardeg}(V, \nabla, \mathcal{F}) = \deg(V) + \sum_{1}^{m} \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\Lambda_i + \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\phi_i$$

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	Dolbeault/Higgs	DR/Connections	Betti/monod.
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eigenvalues $t_{\mathbb{C}}$	$-\frac{1}{2}(\phi+\sigma)$	$\tau+\sigma$	$\exp(2\pi i(\tau+\sigma))$

(eigenvalues of Γ)

$\tau = t_{\mathbb{R}}$, $\sigma = i t_{\mathbb{R}}$ (eigenvalues of Λ)

$$\operatorname{Pardeg}(V,\nabla,\mathcal{F}) = \deg(V) + \sum_{1}^{m} \operatorname{Tr}\theta_i = \sum \operatorname{Tr}\Lambda_i + \operatorname{Tr}\theta_i = \sum \operatorname{Tr}\phi_i$$

Sufficient stability conditions

If no strictly semistable points then $\mathcal{M}$ is complete

1 If (V', ∇') subconnection of (V, ∇)

$$\deg V' = \sum \operatorname{Tr} \Lambda_i' = \sum \operatorname{Tr}(\tau_i' + \sigma_i') \in \mathbb{Z}$$

and $\operatorname{Tr} \tau_i' = \sum_{j \in S} (\tau_i)_{jj}$ for some $S \subset \{1, \dots, \operatorname{rk} V\}$, $\# S = \operatorname{rk} V'$

$$\operatorname{Tr} \sigma_i' = \sum_S (\sigma_i)_{jj}$$

i.e a "subsum" of $\sum_1^m \operatorname{Tr} \tau_i + \operatorname{Tr} \sigma_i$ is in $\mathbb{Z}$

(if $(\underset{\sim}{\tau}, \underset{\sim}{\sigma})$ off of these hyperplanes then $\mathcal{M}$ complete)

Fix $G = GL_n(\mathbb{C})$

(weighted) conjugacy class

$\mathcal{C} \subset \underset{\sim}{H}$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma, \mathcal{C})$$

irregular curve

Hyperkahler manifold

"wild Hitchin space"

(Biquard-B. '04)

Hitchin-Simpson

Biquard-B.

Corlette-Donaldson

Sabbah

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\wr\!\!\parallel$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\wr\!\!\parallel$ irregular RH isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$

(See e.g. survey 1203.6607 for full details)

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$ — Algebraic integrable systems (Hitchin, Nitsure, Bottacin, Markman...)

$\Big|\Big|\Big|$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|$ RH irregular isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

$$\hookrightarrow \mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$
$$\downarrow$$
$$\mathbb{H}$$

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\Big\|\!\Big\wr$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$ —— Isomonodromy systems (as Σ varies in admissible fashion)

$\Big\|\!\Big\wr$ irregular RH isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$

$$\begin{array}{c} \underset{\sim}{\Sigma} \\ \downarrow \\ \mathbb{B} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{M}_{DR}(\Sigma_b) & \subset & \mathcal{M}_{DR/\mathbb{B}} \\ \downarrow & & \downarrow \\ b & \in & \mathbb{B} \end{array}$$

fibre bundle with flat nonlinear connection

e.g. Painlevé equations, Schlesinger system,

JMU system, Simply-laced isomonodromy systems

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$\cong$ wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$

$\cong$ irregular RH isomorphism

$$\mathcal{M}_B(\Sigma, \mathcal{C})$$ — Nonlinear braid/mapping class group actions

"Wild mapping class groups"

$$\begin{array}{c}\Sigma \\ \downarrow \\ \mathbb{B}\end{array} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

e.g. Braiding of Stokes data of Cecotti-Vafa / Dubrovin

Guide to moduli spaces on $\mathbb{P}^1$

Consider open subset $\mathcal{M}^* \subset \mathcal{M}_{DR}$

where bundle / $\mathbb{P}^1$ holomorphically trivial

- often $\mathcal{M}^*$ again has complete hyperkahler metric
- ? view as approximation of more transcendental metric on $\mathcal{M}$
- use known "additive approximations" $\mathcal{M}^*$ as guide to $\mathcal{M}$'s

Classical hyperkahler mfds

1. Complex coadjoint orbits $\quad \mathcal{O} \subset \mathfrak{g}^*$

(Kronheimer, Biquard, Kovalev)

If pole divisor $2(0) + (\infty) \subset \mathbb{P}^1$

have examples where $\quad \mathcal{M}^* \cong \mathcal{O} /\!\!/\!\!/_{\lambda} T_K$

$\left[\mathcal{M}_{Betti} = \mathcal{L} /\!\!/_{\lambda} T, \quad \mathcal{L} \subset G^* \text{ symplectic leaf} \right]$

($T_K \subset T$ compact torus)

② T^*G (Kronheimer)

If pole divisor $2(0) + 2(\infty) \subset \mathbb{P}^1$

have examples where $\mathcal{M}^* \cong T_K \backslash\!\backslash\!\backslash_{\lambda_1} T^*G /\!/\!/_{\lambda_2} T_K$

$$\left[\begin{array}{l} \mathcal{M}_{Betti} = T \backslash\!\backslash_{\lambda_1} \mathbb{D} /\!/_{\lambda_2} T \\ \mathbb{D} \subset (G \times G^*)^2 \quad \text{Lu-Weinstein double sympl. groupoid} \end{array}\right]$$

③ ALE spaces   deformations of $\widetilde{\mathbb{C}^2/\Gamma}$

(Eguchi–Hanson, Gibbons–Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SU_2$ finite $\longleftrightarrow$ ADE affine Dynkin graph

logarithmic realisations   only irregular realisations

Fact In cases E_8, E_7, E_6, D_4, A_3, A_2, A_1

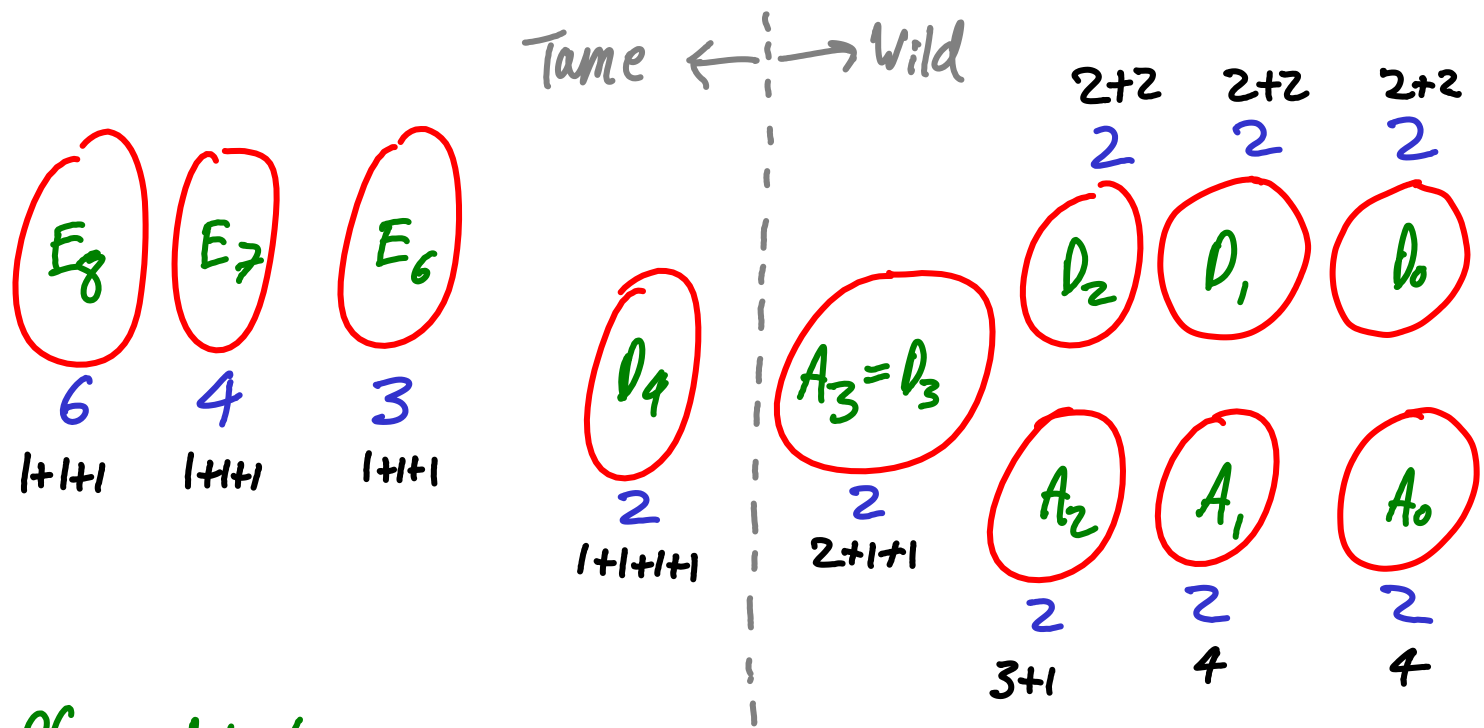
have $\mathcal{M}$ s.t $\mathcal{M}^* \subset \mathcal{M}$ is corresponding ALE space

	Pole orders
A_3	$2+1+1$
A_2	$3+1$
A_1	4

– Okamoto found in 1987 the corresponding affine Weyl groups are the sym gps of the corresponding Painlevé equations

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces)



affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

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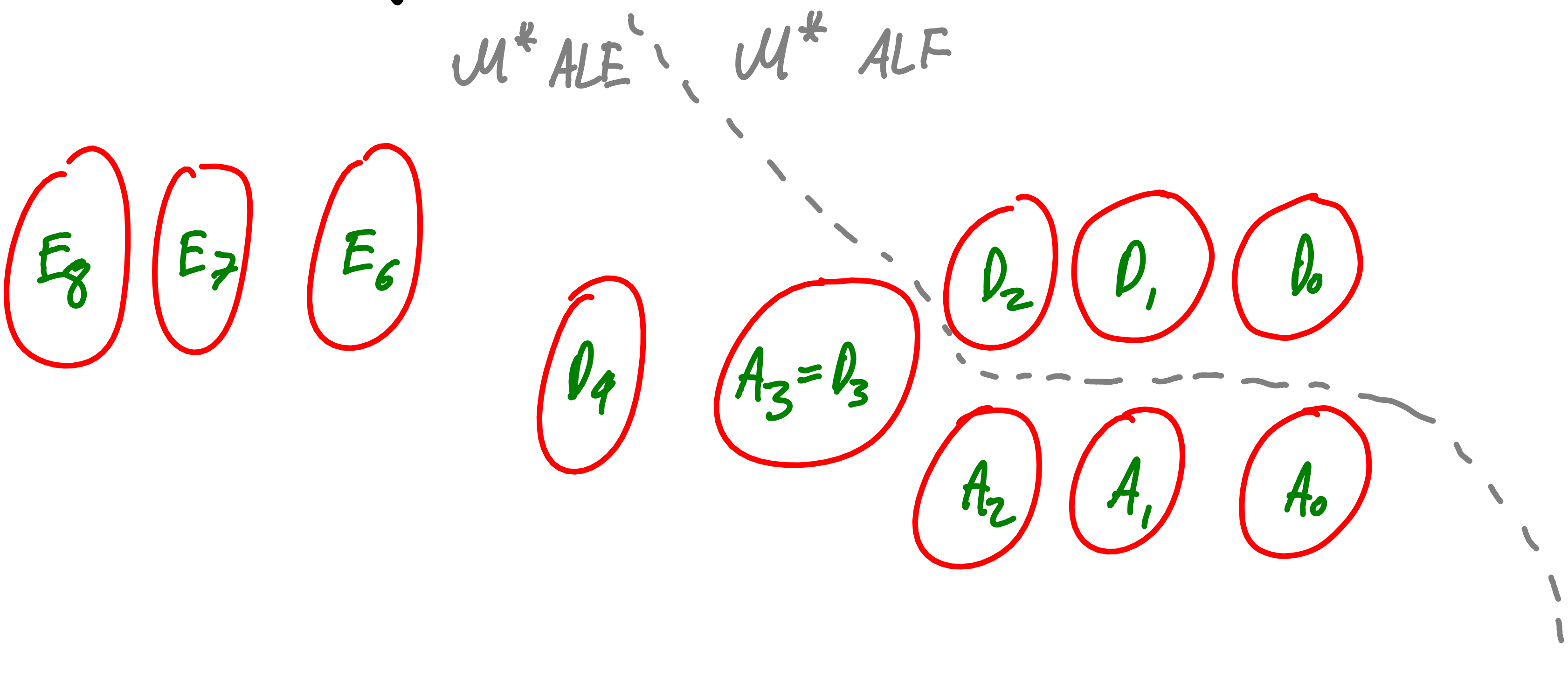

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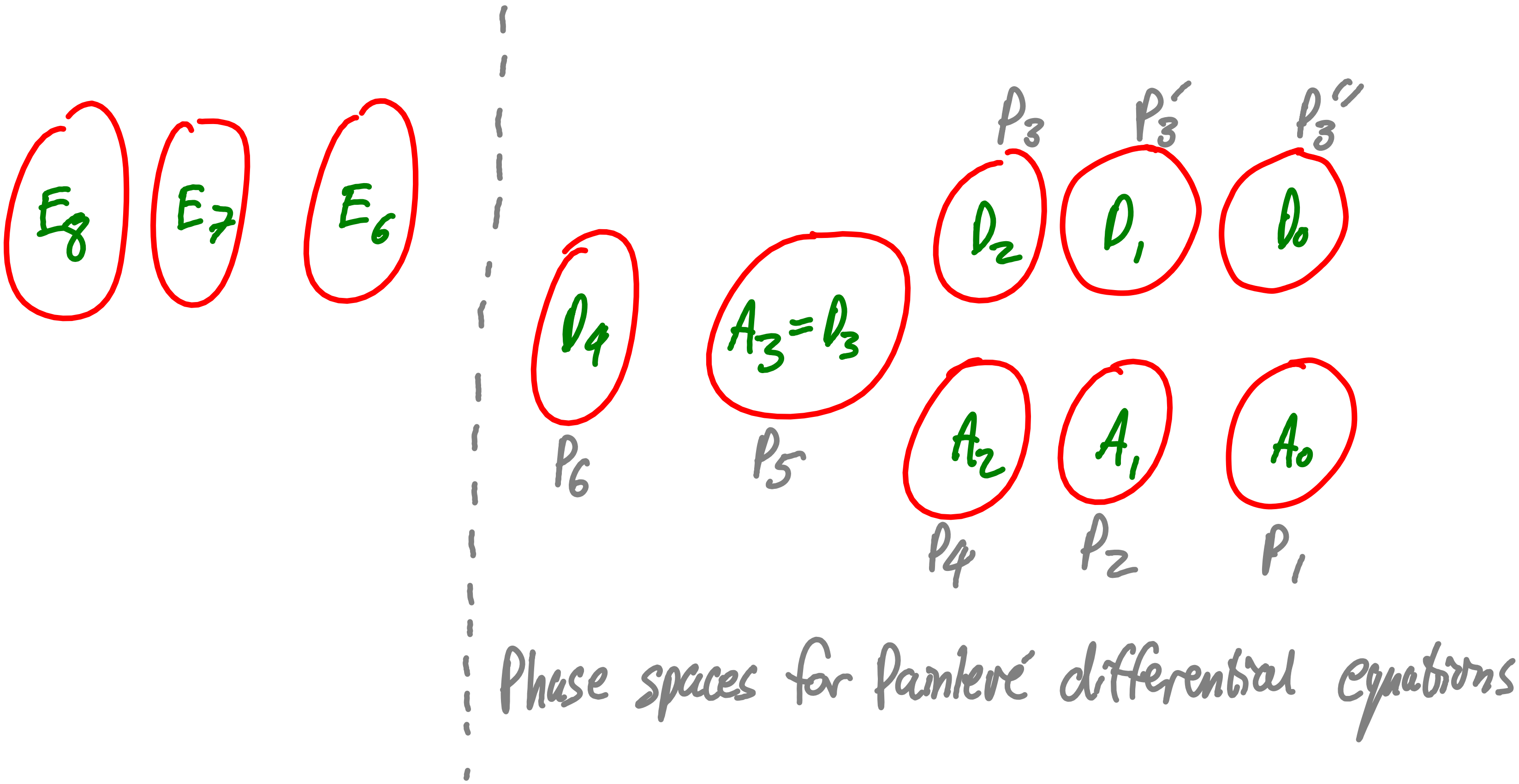
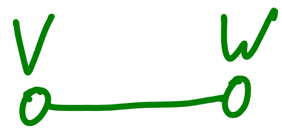


4 (Nakajima) Quiver varieties



$\mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V)$ is hyperkahler $U(V) \times U(W)$ space

Graph = ADE dynkin graph $\Rightarrow$ ALE space (Kronheimer)

else in general get higher dim^n hyperkahler mfd (or empty)

— lets consider simply-laced cases

E.g. Fuchsian case $\quad G = GL_n(\mathbb{C})$

$$\mathcal{M}^* \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G$$

$(\mathcal{O}_i \subset \mathfrak{g}^*$ coadjoint orbits$)$

point of $\mathcal{M}^*$ $\sim$ Fuchsian system $\displaystyle\sum_1^m \frac{A_i}{z-a_i}\,dz$ $\quad A_i \in \mathcal{O}_i$, $\sum A_i = 0$

E.g. Fuchsian case $\quad G = GL_n(\mathbb{C})$

$$\mathcal{M}^* \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G$$

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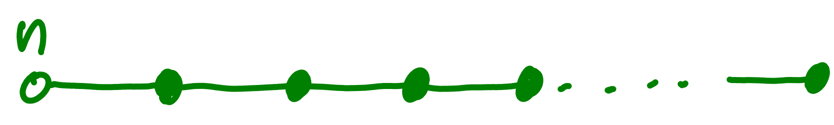
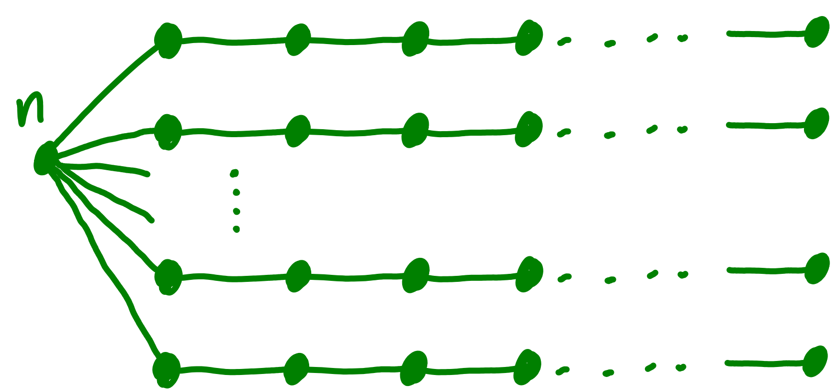
Relation to quivers (Kraft-Procesi, Nakajima, ...)

$\mathcal{O}_i \cong$

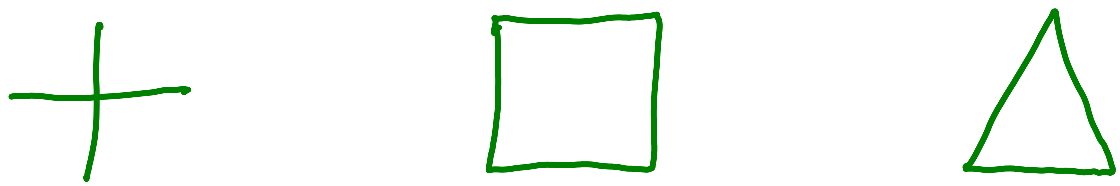


$\mathcal{M}^* \cong$

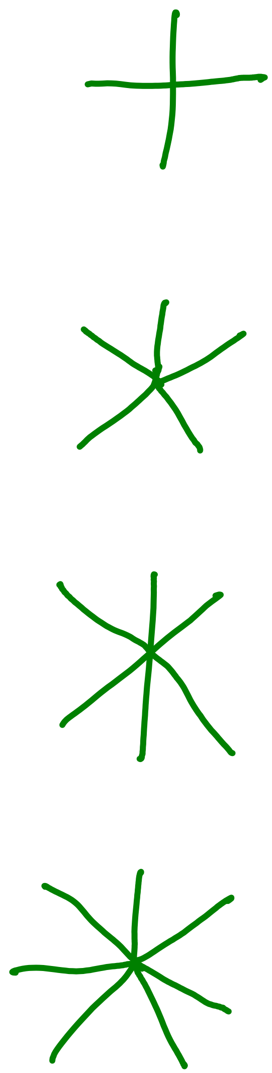


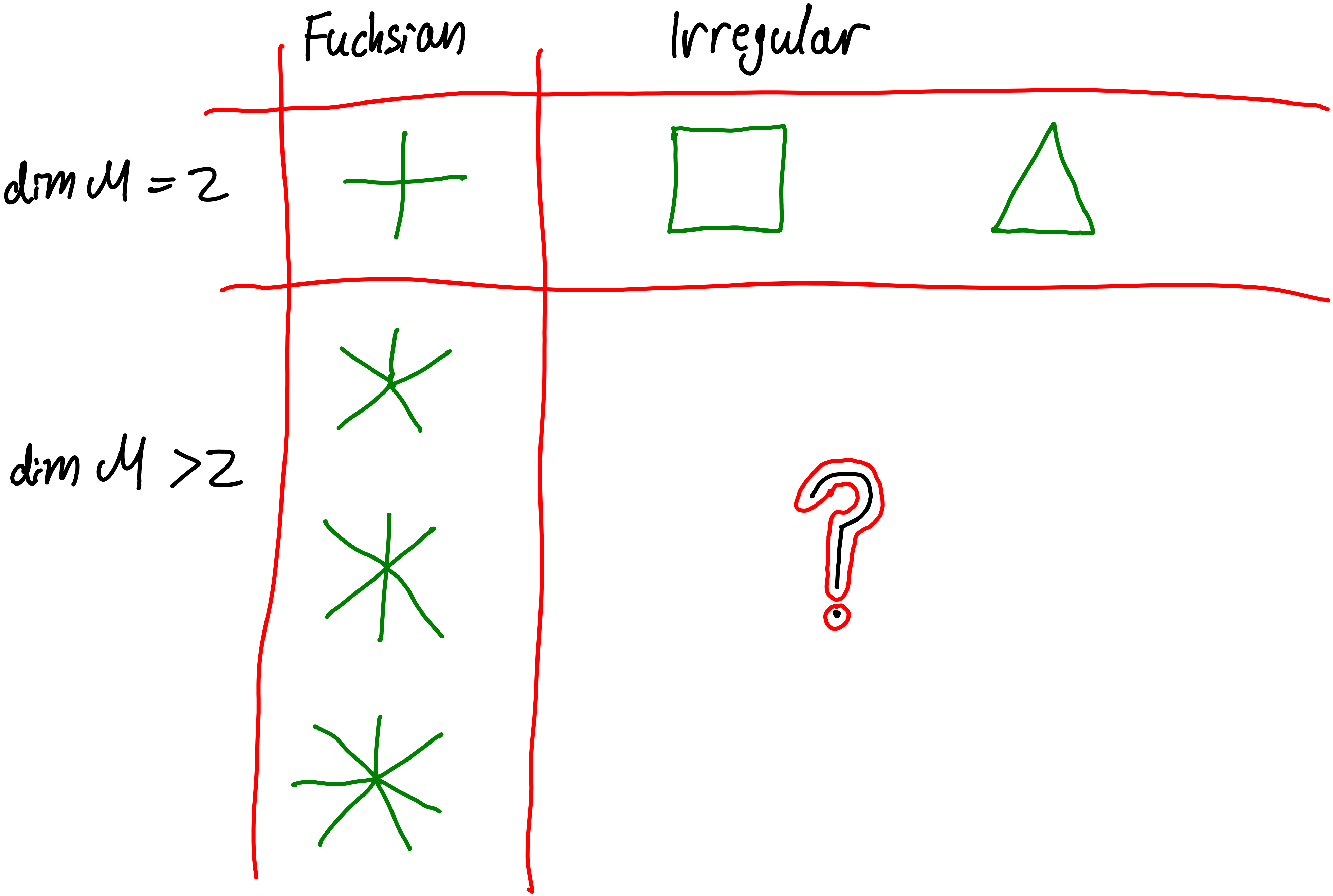
"Starshaped" quivers used by Crawley-Boevey in Deligne-Simpson problem

Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type A_2, A_3, D_4 resp.



Recall Crawley-Boevey related moduli spaces of Fuchsian systems to star-shaped quivers (building on Kraft-Procesi, Nakajima, ...)

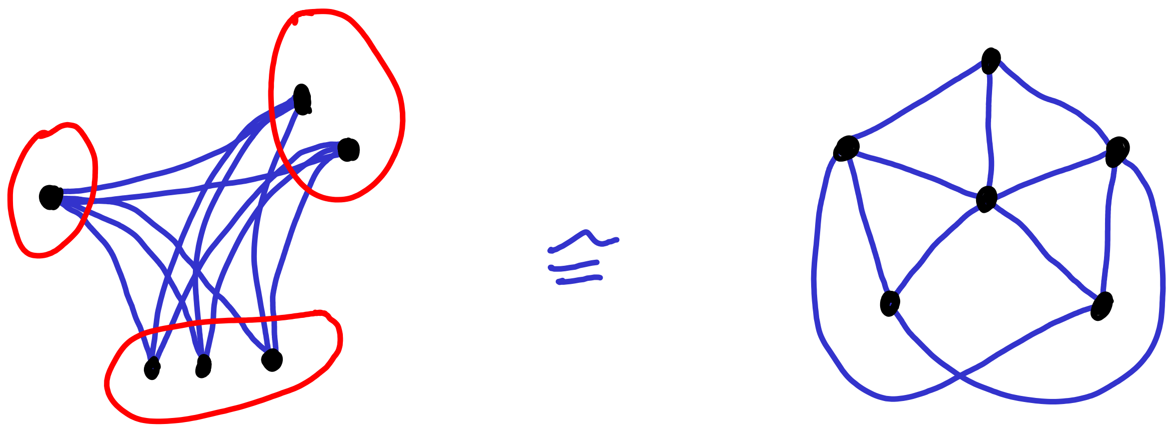



Fuchsian
Irregular
dim M = 2
dim M > 2

Thm

Can take *any* complete k-partite graph (for any k)

E.g.



$$\Gamma(3,2,1)$$

– get action of corresponding (not necessarily affine) Kac-Moody Weyl group

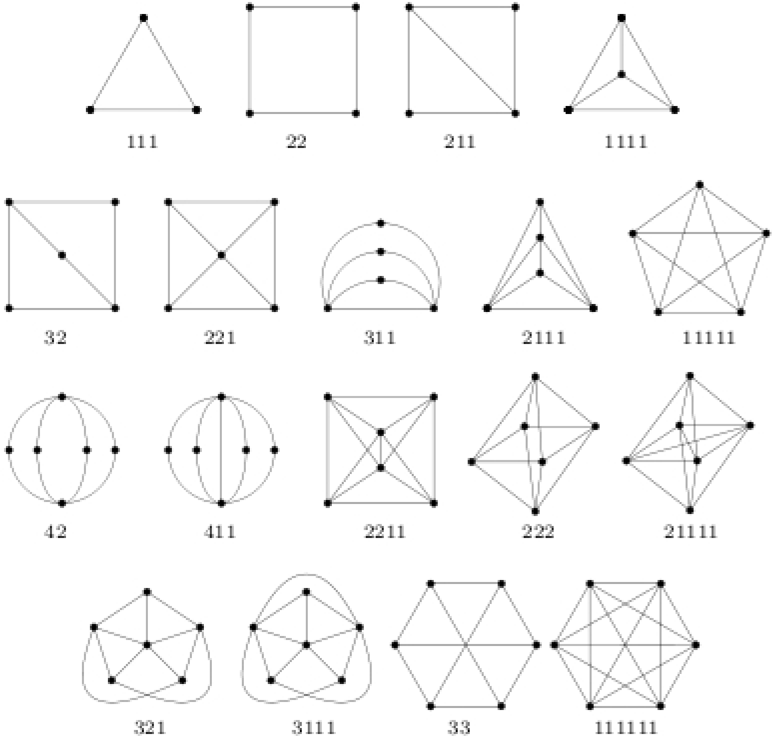


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n,1)$ and the totally disconnected graphs $\Gamma(n)$)

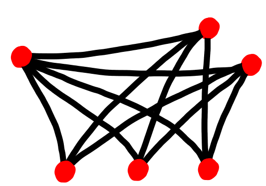
More general and precise statement:

Definition A graph Γ is a

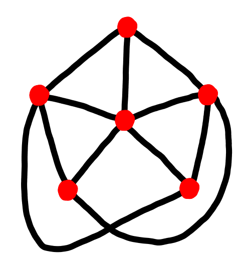
- "nonabelian Hodge graph" if there is some (rational) irregular curve Σ s.t.

$$\begin{array}{c}\mathcal{M}^*(\Sigma) \cong \text{a quiver variety attached to } \Gamma \\ \cap \\ \mathcal{M}(\Sigma)\end{array}$$

- "supernova graph" if obtained by gluing some legs onto a complete k-partite graph



$\cong$



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s.t. $\mathcal{M}^*(\Sigma) \cong$ a quiver variety attached to Γ

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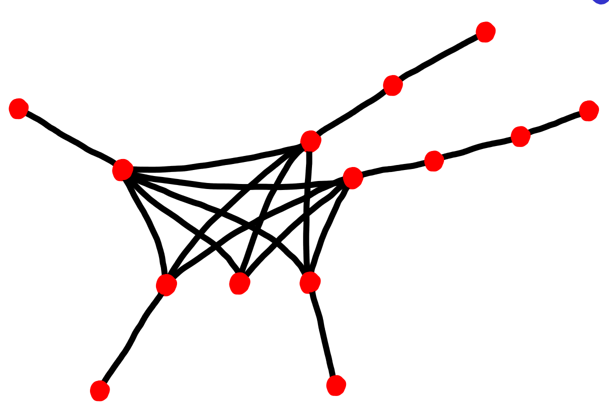
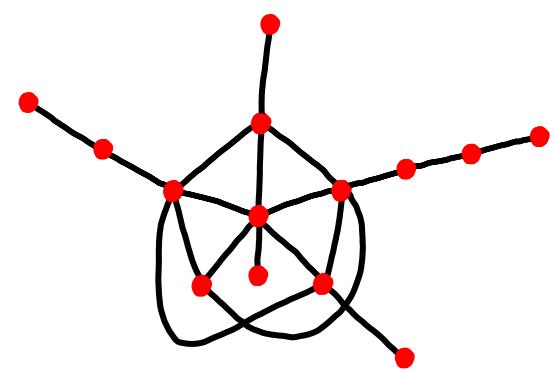


$\cong$



— generalising the star-shaped graphs

Thm

Any supernova graph is a nonabelian Hodge graph

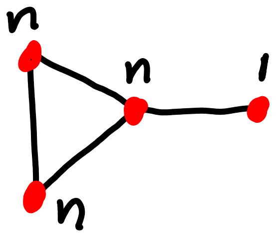
so can attach nonabelian Hodge structure $\mathcal{M}$ to any such graph

& thus

- a Hitchin system
- an isomonodromy system

Moreover Γ determines a (symmetric) Kac-Moody root system & Weyl group, and Weyl group elements lift to give isomorphisms between such systems

E.g.: Higher/hyperbolic/Hilbert Painlevé systems

$\Gamma_n =$ 

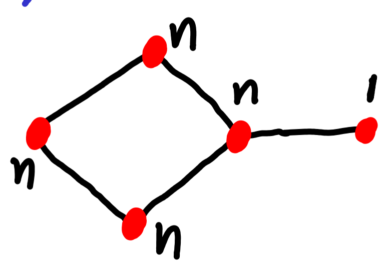
$\Longrightarrow \quad hP_{IV}^n := \mathcal{M}(\Gamma_n)$ dimension $2n$

$n=1 \qquad hP_{IV}^1 \cong P_{IV}$ dim 2

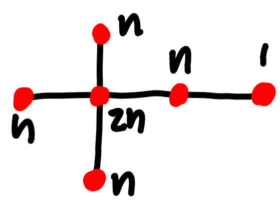
$$\mathcal{M}^*(\Gamma_n) \underset{\text{diffeo}}{\cong} \mathrm{Hilb}^n(\mathcal{M}^*(\Gamma_1))$$

Question: $\mathcal{M}(\Gamma_n) \overset{?}{\cong} \mathrm{Hilb}^n(\mathcal{M}(\Gamma_1))$ (for generic parameters)

similarly for any 2d Hitchin system e.g.:

$\Gamma_n =$ 

$\Longrightarrow \quad hP_{V}^n := \mathcal{M}(\Gamma_n)$ dimension $2n$

$\Gamma_n =$ 

$\Longrightarrow \quad hP_{VI}^n := \mathcal{M}(\Gamma_n)$ dimension $2n$