Higgs bundles and fixed points

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UIUC

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Outline

- Some Higgs bundle background
- Examples
- Fixed points
- Examples and an application

Setting

- Σ a closed Riemann surface of genus ≥ 2 with canonical bundle K.

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Example

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Corlette's Theorem for Harmonic metrics on flat bundles

$$\tilde{\Sigma} \to SL(n, \mathbb{C})/SU(n)$$

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Can do the same thing for $PSP(2n,\mathbb{R})$.

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As a $SL(2n, \mathbb{C})$ bundle $\left(V \oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$.

Hitchin's parameterization of Hitchin Component

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$(\mathcal{E}, \phi) \cong (\mathcal{E}, \lambda\phi)$ for all $\lambda \in U(1)$, (K-Twisted) Holomorphic chain:

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Lemma

If $\lambda \in U(1)$ and g_λ acts by $(\mathcal{E}, \phi) \mapsto (\mathcal{E}, \lambda\phi)$ then g_λ is unitary.

Theorem (C.)

Let $(\mathcal{E},\phi)$ be a stable $SL(n,\mathbb{C})$-Higgs bundle and $\zeta_k = e^{\frac{2\pi i}{k}}$ then $(\mathcal{E},\phi) \cong (\mathcal{E},\zeta_k\phi)$ if and only if

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Proof

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Examples of Fixed points

As a twisted quiver bundle

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with $q_j = 0$ for $j \neq 0 \ mod \ k$.

Theorem (C.)

Let (V, β, γ) be a $SP(2n, \mathbb{R})$-Higgs bundle that is stable and simple then $(V, \zeta_k\beta, \zeta_k\gamma) \cong (V, \beta, \gamma)$ if and only if (V, β, γ) is fixed by all of $U(1)$ or

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Furthermore, the metric splits as $H_1 \oplus \cdots \oplus H_k$ and satisfies

$$det(H) = 1 \qquad \text{and} \qquad H^T Q H = Q.$$

$\langle \zeta_{n-1} \rangle$ Fixed points in the Hitchin component

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An application or two

Simplification of Equations and Asymptotics

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Let (E, Q, ϕ) *in the Hitchin component be a fixed point of* $\langle \zeta_n \rangle$ *or* $\langle \zeta_{n-1} \rangle$ *then*

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$$\phi^* = \begin{pmatrix} 0 & h_1^{-1}h_2 & & & \\ & & h_2^{-1}h_3 & & \\ & & & \ddots & \\ h_1h_2\bar{q}_{n-1} & & & 0 & h_1^{-1}h_2 \\ 0 & h_1h_2\bar{q}_{n-1} & & & 0 \end{pmatrix} \qquad \phi^* = \begin{pmatrix} 0 & h_1^{-1}h_2 & & & \\ & & h_2^{-1}h_3 & & \\ & & & \ddots & \\ & & & 0 & h_1^{-1}h_2 \\ h_1^2\bar{q}_n & & & & 0 \end{pmatrix}$$

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Metric Theorem

Theorem (C.-Li)

Let $(\mathcal{E}, \phi)$ be a Higgs bundle in the $SL(n, \mathbb{R})$-Hitchin component with

$$\phi = \tilde{e}_1 + tq_{n-1}e_{n-2} \qquad \text{or} \qquad \phi = \tilde{e}_1 + tq_n e_{n-1},$$

for $t \in \mathbb{R}$. Then, for all $p \in \Sigma$ away from the zeros of q_{n-1} or q_n, as $t \to \infty$, the metric solving the Higgs bundle equations has the following form:

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Baraglia used this in his thesis to give description of $SP(4, \mathbb{R})$-Hitchin component in terms of 4-cyclic Higgs bundles.

Thank you