A dilogarithm identity on the moduli space of curves

Feng Luo

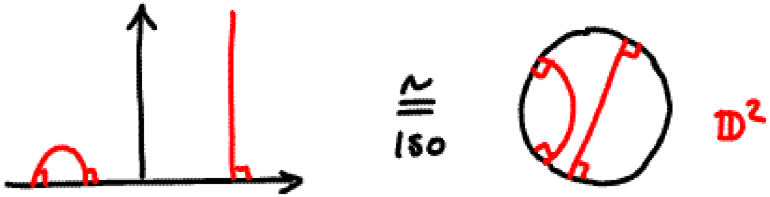
Rutgers University

Joint work with Ser-Peow Tan

GEOMETRY, QUANTUM TOPOLOGY AND ASYMPTOTICS

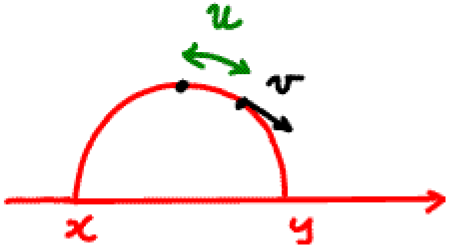
Singapore, August 7, 2014

- hyperbolic plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$



- $\mathrm{Iso}^+(\mathbb{H}) = PSL(2.\mathbb{R}) \cong UT\mathbb{H}$ unit tangent bundle

- Haar measure $\dfrac{2\,dx\,dy\,du}{(x-y)^2}$

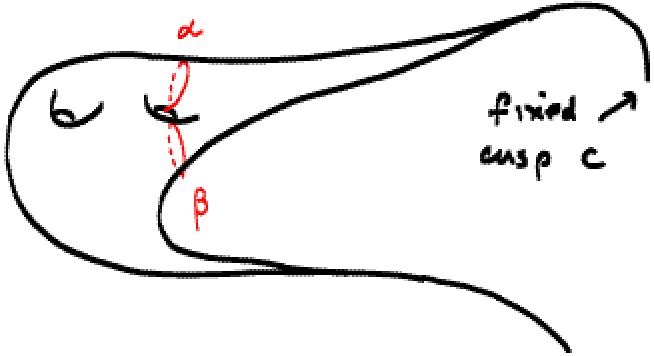


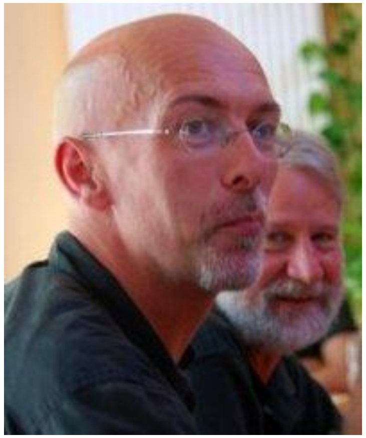
- Uniformization theorem Σ Riemann surf, $\chi(\Sigma) < 0 \Rightarrow \Sigma \cong \mathbb{H}/\Gamma$

$UT\Sigma \cong UT\mathbb{H}/\Gamma \cong PSL(2,\mathbb{R})/\Gamma$

$\Rightarrow$ $UT\Sigma$ has volume form vol.

- McShane (1998), **Σ** hyperbolic , n>0 cusps

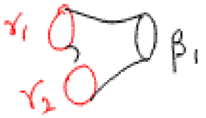


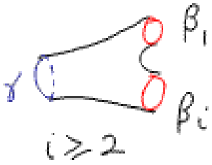


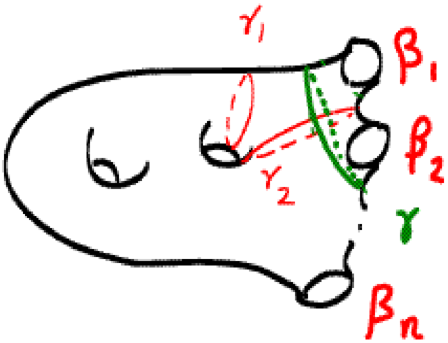
$$\sum \frac{1}{1+e^{\frac{1}{2}|\alpha|+\frac{1}{2}|\beta|}} = \frac{1}{2}$$

- Mirzakhani (2007), Tan-Wong-Zhang (2006), Σ hyperbolic , n>0 geodesic boundary

$$\sum \mathcal{D}(|\beta_1|, |\gamma_1|, |\gamma_2|) + \sum \mathcal{R}(|\beta_1|, |\beta_i|, |\gamma|) = |\beta_1|$$

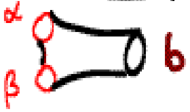


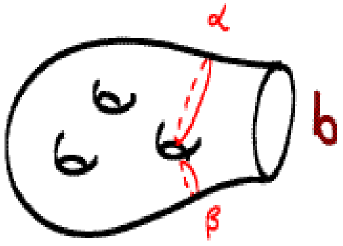




$n=1$

$$\sum 2\tanh^{-1}\left(\frac{\sinh(|b|/2)}{\cosh(|b|/2)+e^{\frac{1}{2}|\alpha|+\frac{1}{2}|\beta|}}\right) = \frac{|b|}{2}$$





$\Downarrow$

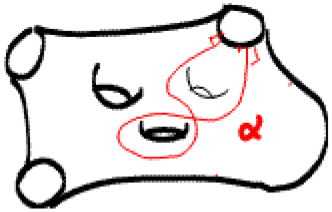
$$\sum \frac{1}{1+e^{\frac{1}{2}|\alpha|+\frac{1}{2}|\beta|}} = \frac{1}{2}$$

Mirzakhani proved important results on Mod(g,r) using her identity:

computing volume of Mod(g,r), etc

- Bridgeman (2009), Σ hyperbolic, ∂Σ geodesic boundary,

$$8\sum_{\alpha}\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{|\alpha|}{2}\right)}\right) = -4\pi^2\chi(\Sigma)$$

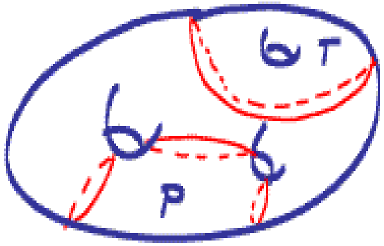


α – orthogeodesic from $\partial\Sigma$ to $\partial\Sigma$.

dilogarithm $\mathcal{L}(0)=0,\qquad \mathcal{L}'(x) = -\frac{1}{2}\left[\frac{\ln(1-x)}{x}+\frac{\ln(x)}{1-x}\right]$

Q: Are there McShane type identities for closed hyperbolic surfaces?

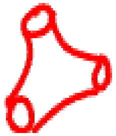
Thm 1. (L-Tan) For any hyperbolic closed surfaces Σ_g



$$\sum_{P} f(p) + \sum_{T} g(T) = -4\pi^2 \chi(\Sigma_g)$$

P:  T: 

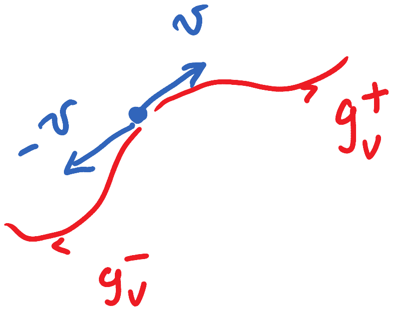
where f, g are diloganthms on the lengths of s.c. geodesics in P, T ,

P =  , T =  geodesic boundary in Σ_g

RM. It also holds for hyperbolic ∑ with geodesic boundary and/or cusp ends.

- UTΣ = unit tangent bundle of Σ

- $v \in UT\Sigma$ generates geodesic rays from v, -v

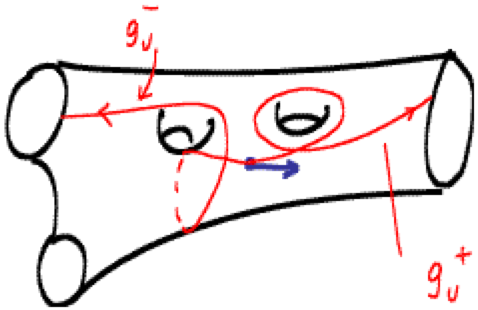


- vol = invariant measure on UTΣ, s.t. vol(UTΣ) = 2π Area(Σ)

Bridgeman's work, assume $\partial\Sigma \neq \phi$.

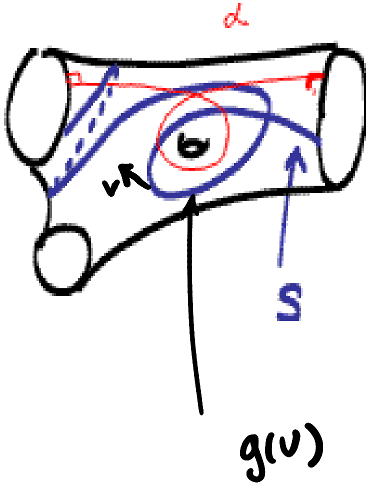
Ergodicity of geodesic flow:

for a.e. $v \in UT\Sigma$, both g_v^+ and g_v^- end at $\partial\Sigma$.



Lemma 1. A geodesic s from ∂Σ to ∂Σ is homotopic to a unique orthogeodesic α.

$$UT\Sigma = \{\text{measure zero}\} \sqcup \bigsqcup_{\alpha\text{- orthogeod.}} B(\alpha)$$



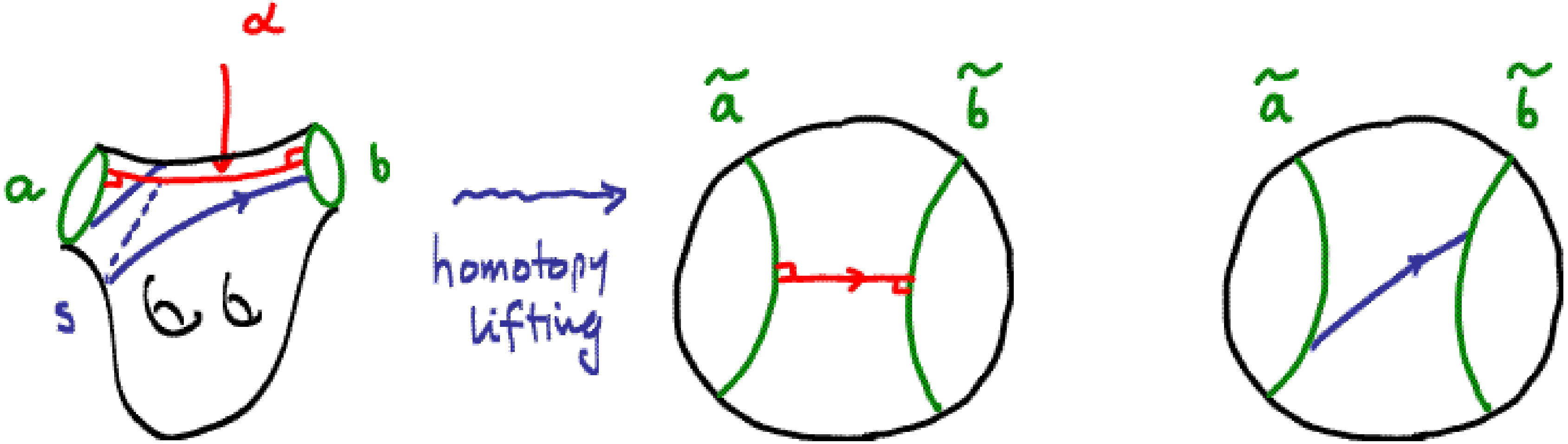
$$B(\alpha) = \{ v \in UT\Sigma \mid g(v) \simeq \alpha \}$$

Bridgeman's identity:

$$\mathrm{vol}(UT\Sigma) = \sum_{\alpha} \mathrm{vol}(B(\alpha)).$$

What is vol(B(α))?

Bridgeman’s calculation of B(α)



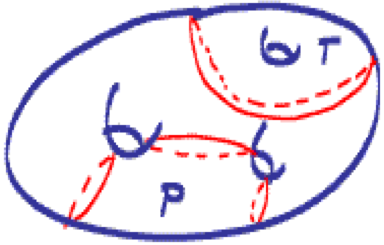
$$B(\alpha) \cong \{ v \in UT\mathbb{H}^2 \mid g_v^+ \cap (\tilde{a} \cup \tilde{b}) \neq \phi , \; g_v^- \cap (\tilde{a} \cup \tilde{b}) \neq \phi \}$$

Bridgeman's Thm $\quad \mathrm{Vol}(B(\alpha)) = 8 \, \mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{\ell(\alpha)}{2}\right)} \right)$

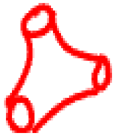
Back to

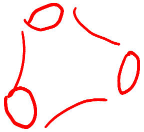
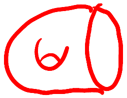
Thm 1. (L-Tan) For any hyperbolic closed surfaces Σ_g

$$\sum_{P:} f(p) + \sum_{T:} g(T) = -4\pi^2\chi(\Sigma_g)$$



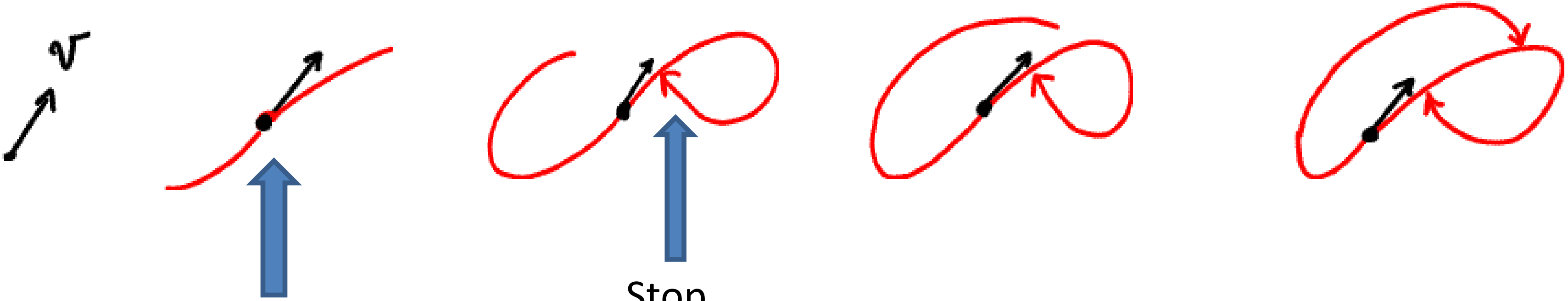
where f, g are diloganthms on the lengths of s.c. geodesics in P, T ,

$P =$  , $T =$  geodesic boundary in Σ_g

Decompose $UT\Sigma$ according to $P =$  and $T =$ 

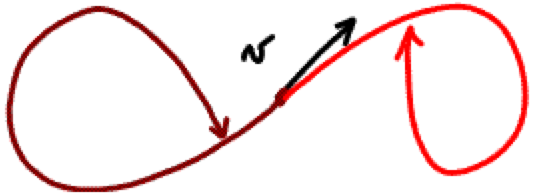
Assume $\partial\Sigma = \phi$. Ergodicity of geodesic flow:

for a.e. $v \in UT\Sigma$, both g_v^+ and g_v^- are not simple.



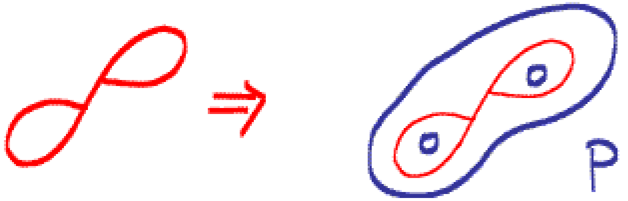
Stop

equal speed in both directions

Define $G(v) =$ 

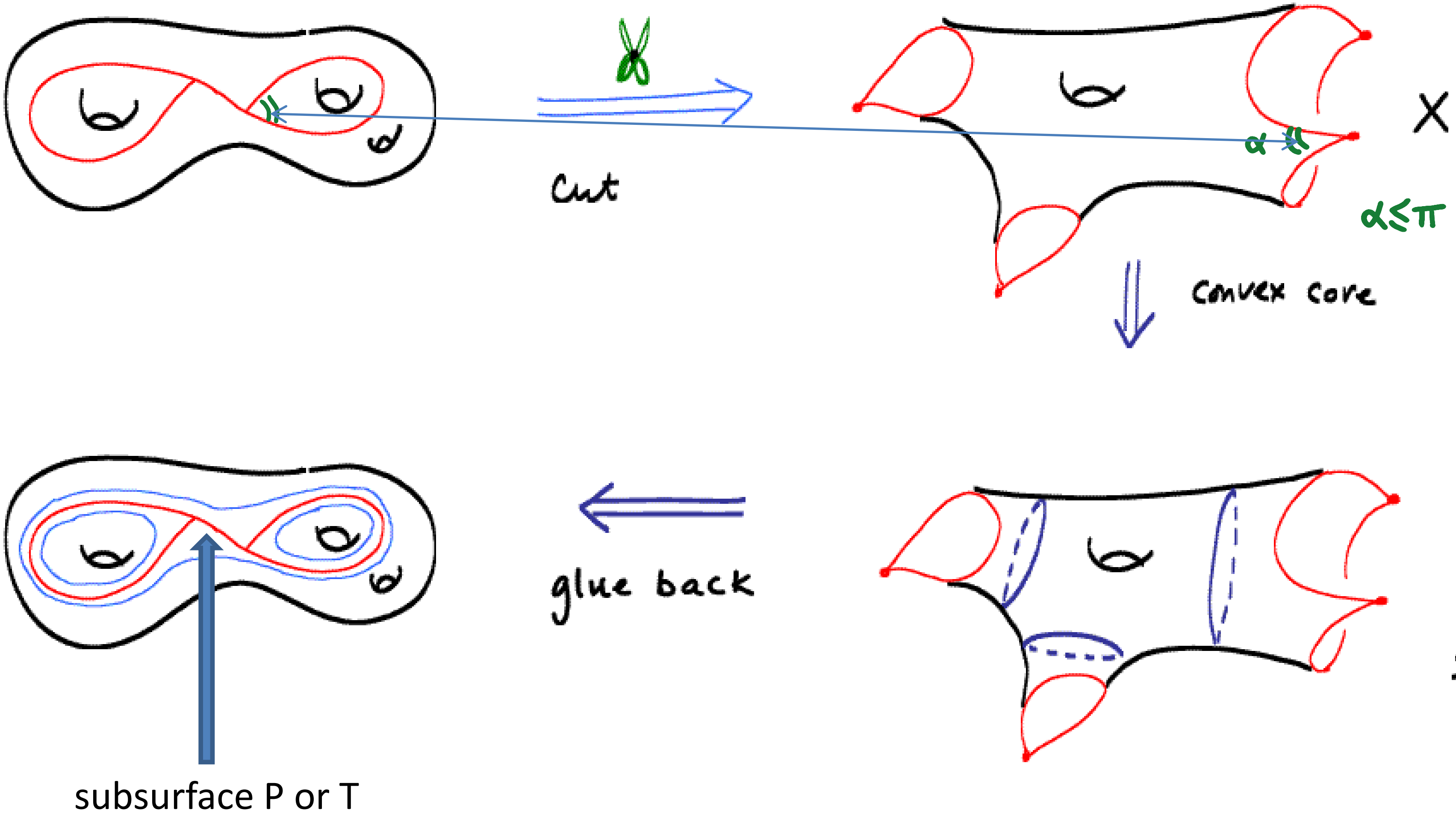
$\simeq \infty$

Lemma 2. $\exists !$ $P =$  or $T =$  in Σ so that $G(v) \subset P$ or T and ∂P, ∂T are geodesics.



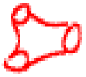
,

Pf: Cut Σ open along $G(v)$ $\Longrightarrow$ get cpt surface X w/ convex ∂X



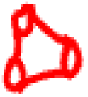
□

$$\Rightarrow \quad UT\Sigma = \{\text{measure zero}\} \cup \bigsqcup_{P} W(P) \sqcup \bigsqcup_{T} W(T)$$

$P =$  $T =$ 

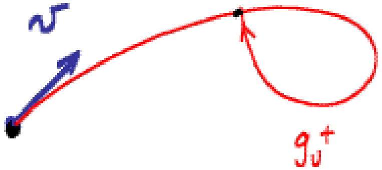
where $W(P) = \{ v \in UT\Sigma \mid G(v) \simeq P \}$, $W(T)$ same

Thus $$-4\pi^2 \chi(\Sigma) = \sum_{P} \mathrm{vol}(W(P)) + \sum_{T} \mathrm{vol}(W(T))$$

P  T 

<u>Q</u>: $\mathrm{vol}(W(P))$, $\mathrm{vol}(W(T))$?

Open problem Σ closed hyperbolic, a.e., $v \in UT\Sigma$, g_v^+ Not simple



$\simeq$ simple closed geod γ

$$UT\Sigma = \{ \text{measure zero} \} \cup \bigsqcup_{\gamma \text{ s.c. geod}} \Omega(\gamma)$$

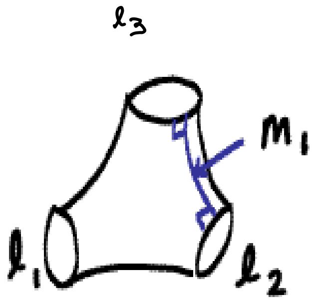
$$\Omega(\gamma) = \{ v \in UT\Sigma \mid g_v^+[0,t_1] \simeq \gamma \}$$

$$-4\pi^2\chi(\Sigma) = \sum_{\gamma \text{ s.c.}} \mathrm{vol}(\Omega(\gamma))$$

We don’t know how to compute this volume.

Thm 2. (Tan-L)

P :



$x_i = e^{-l_i}$, $y_i = \tanh^2\left(\frac{m_i}{2}\right)$.

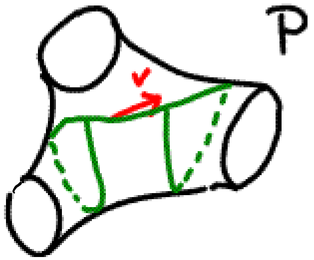
$$\mathrm{vol}(W(P)) = 4\sum_{i\neq j}\left[2\mathcal{L}\left(\frac{1-x_i}{1-x_iy_j}\right) - 2\mathcal{L}\left(\frac{1-y_j}{1-x_iy_j}\right) - \mathcal{L}(y_j) - \mathcal{L}\left(\frac{(1-y_j)^2x_i}{(1-x_i)^2y_j}\right)\right]$$

Pentagon Relation

$$\mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) + \mathcal{L}(y) + \mathcal{L}(x) + \mathcal{L}(1-xy) = \frac{\pi^2}{2}$$

There is a similar formula for vol(W(T)).

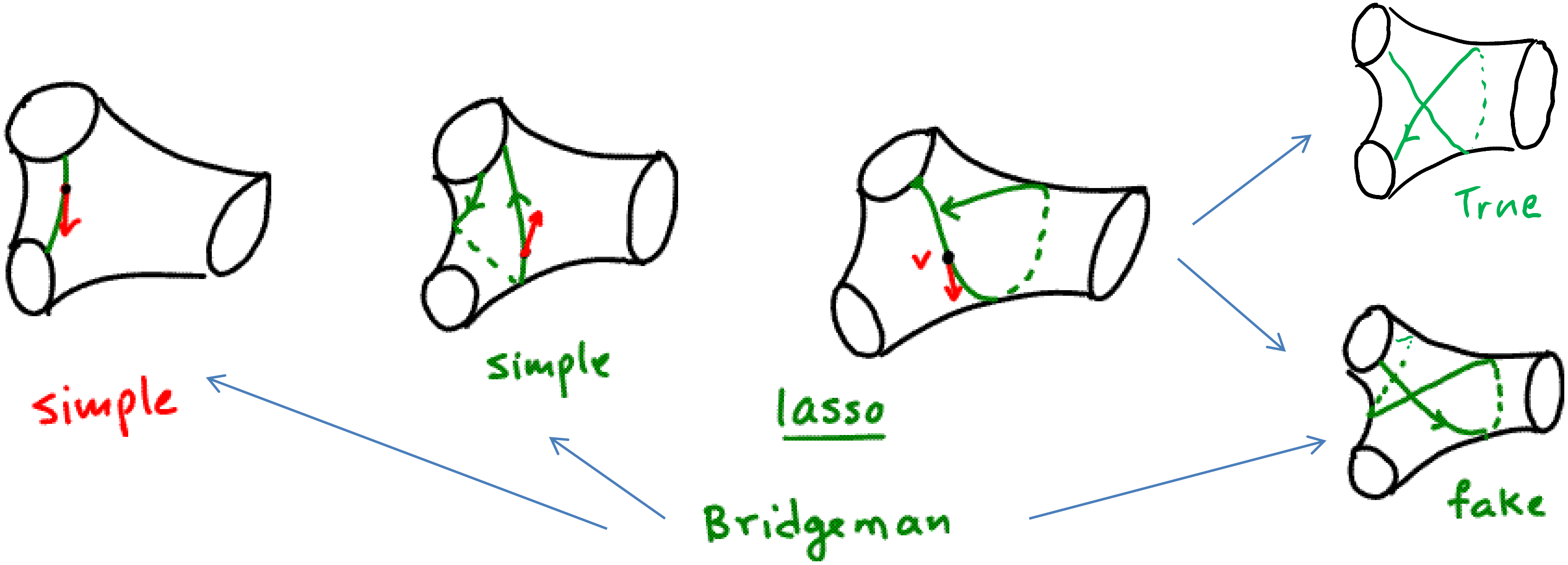
How to compute vol(W), $W = \{ v \in UTP \mid G(v) \approx P \}$



First $\mathrm{vol}(UTP) = 4\pi^2$

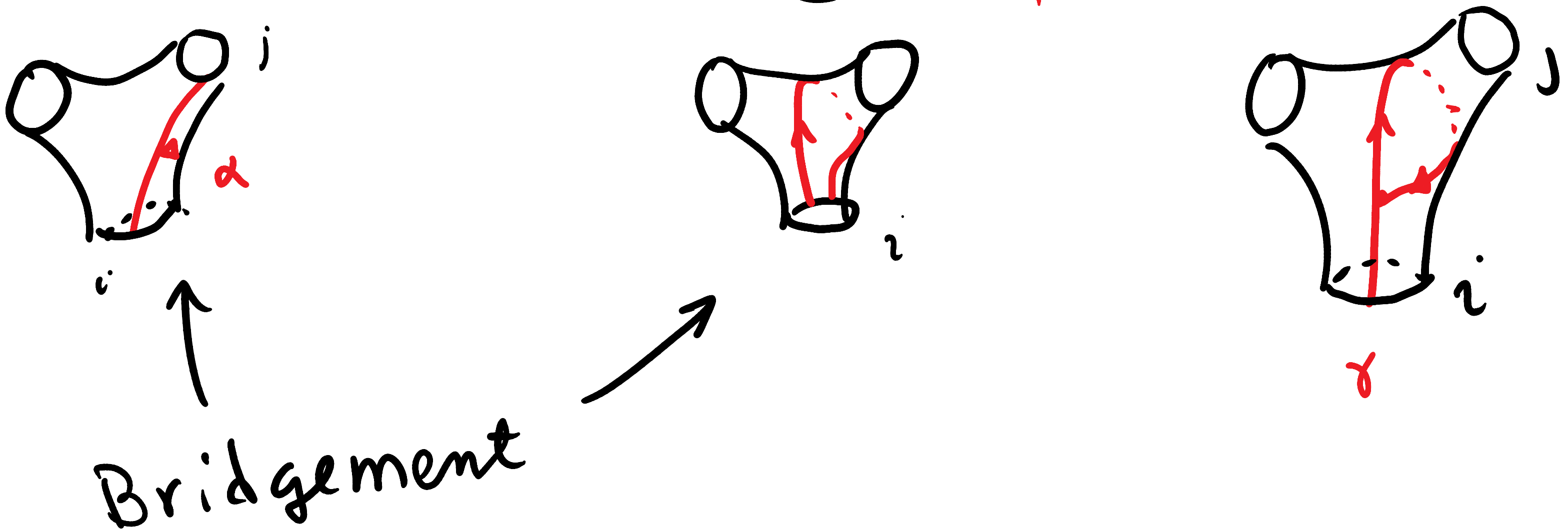
Consider $UTP - W$.

So $v \in UTP - W \iff$



UTP-W =

$$\bigcup_{\Omega(i,j)} \bigcup \bigcup_{\Omega(\beta(i))} \bigcup \bigcup_{\mathcal{U}(i,j)}$$

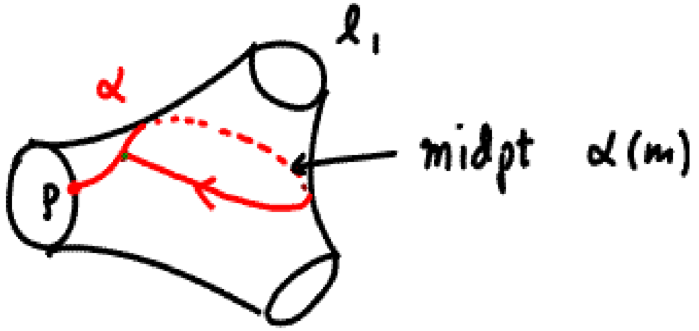


Def. A lasso is a geodesic $\alpha : [0, T] \longrightarrow P$ s.t

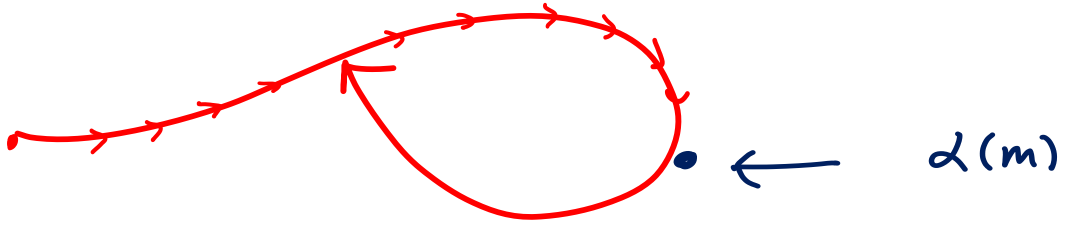
(1) $\alpha(0) \in \partial P$

(2) $\alpha|_{[0,T)}$ 1-1

(3) $\alpha|_{[0,T]}$ Not 1-1



For $0 \leq t \leq m$, $v = \alpha'(t)$ said to generate α

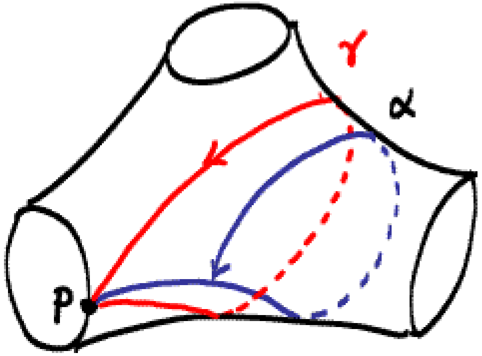


Question : $\mathrm{vol}\{ v \in UTP \mid v \text{ generates a lasso} \}$

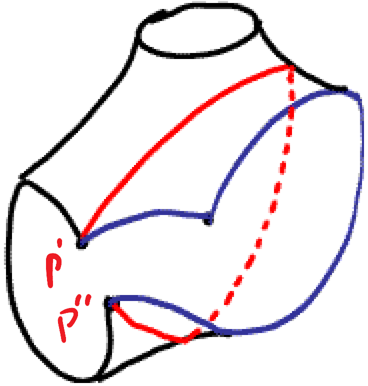
Lemma 3 α lasso from p. Then $\exists$ geodesic loop γ s.t

① $\alpha \cap \gamma = \{p\}$.

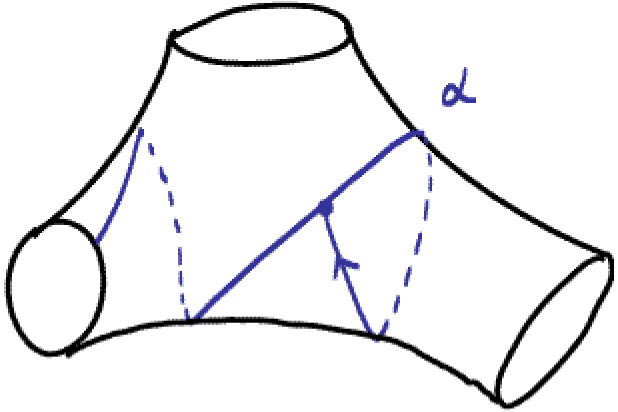
② $\gamma \simeq \alpha$.



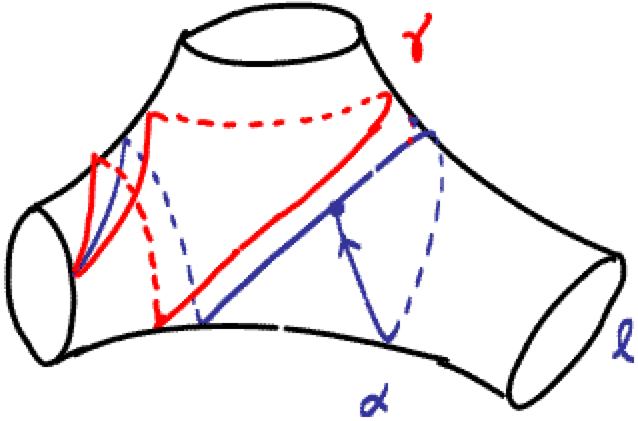
pf: Cut P open along α to obtain X w/ convex boundary.



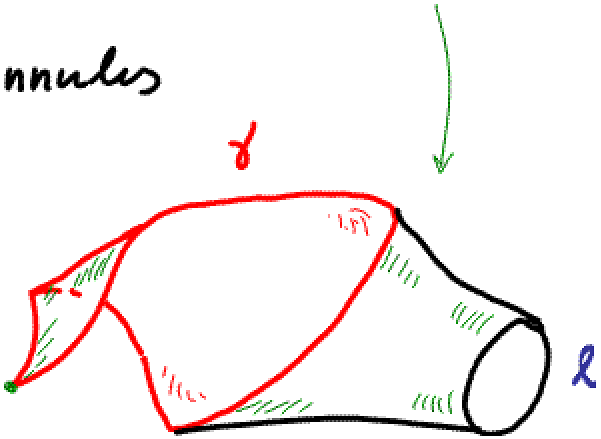
γ shortest path from p' to p'' in X.



lasso

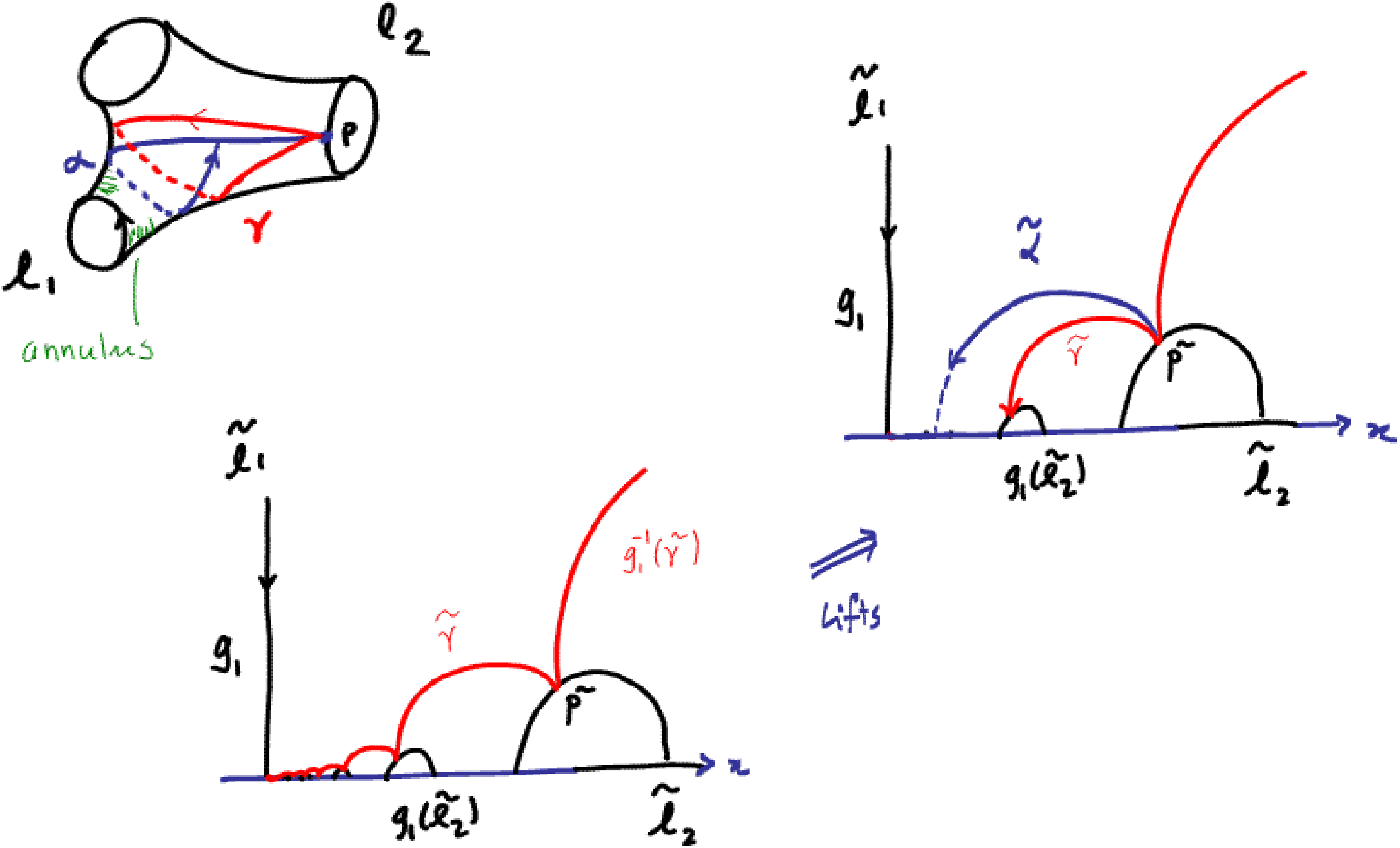


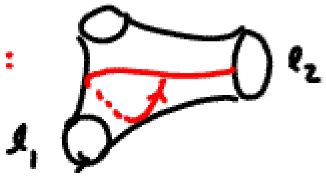
∃ convex annulus



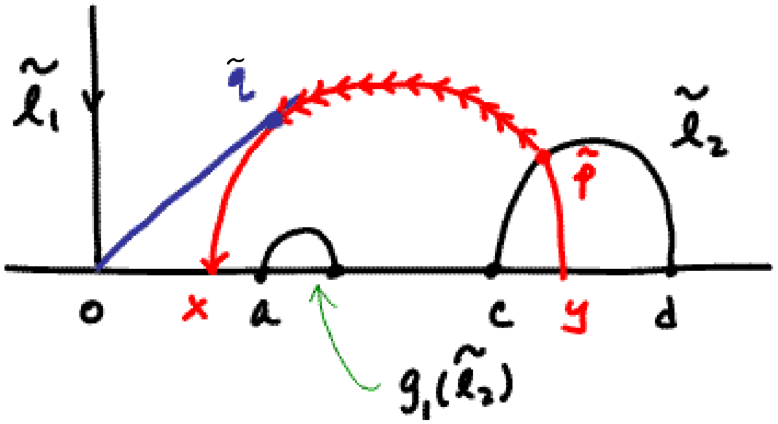
geodesic in it from p

Consider all v generating lassos α from l_2 homotopic to l_1



Conclusion $\{ v \in UTP \mid v$ generates α : 

$\}$

$\cong \{ v \in UT\mathbb{H} \mid$ 

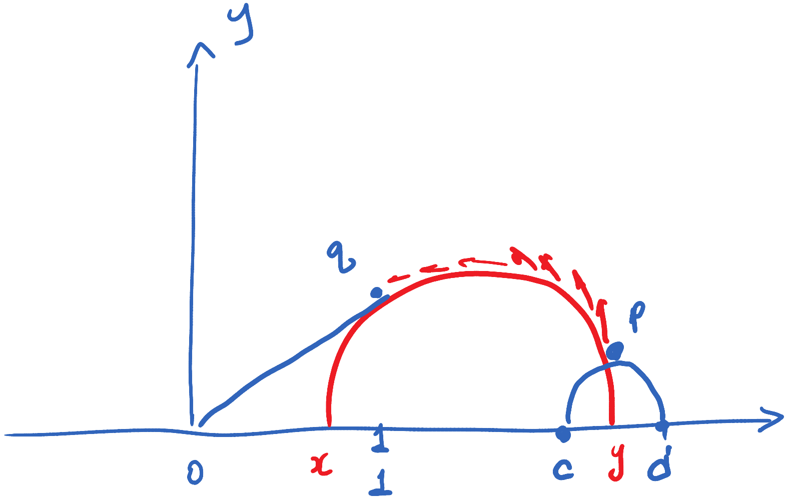
$, \quad 0 < x < a, \quad c < y < d \}$

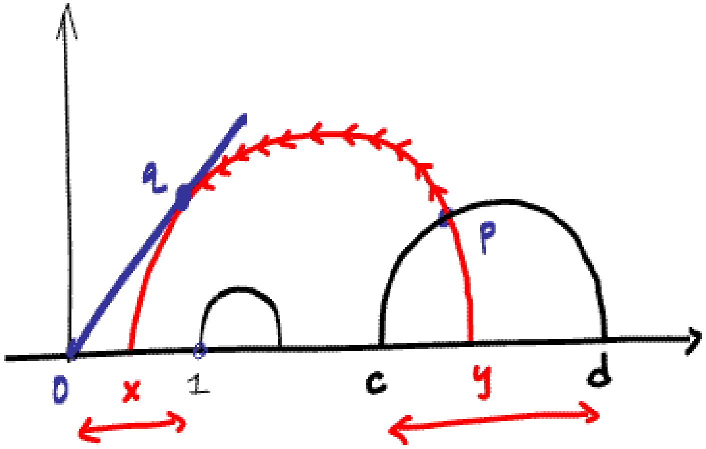
$= \{ v \in UT\mathbb{H} \mid v \in \mathrm{Geod}[y, x], \ 0 < x < a, \ c < y < d,$
v between $\hat{p}, \tilde{q} \}$

Prop. Fix $0<1<c<d$, the volume of

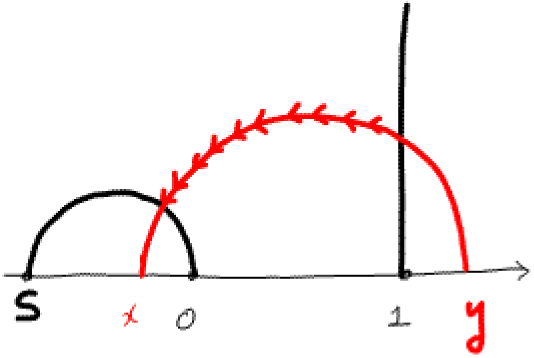
$$\{ v \in UT\mathbb{H}^2 \mid G(v) = [x, y],\ 0<x<1,\ c<y<d,\ v \text{ between } p, q \}$$

is
$$\int_0^1 \int_c^d \frac{\ln\left|\frac{y(x-c)(x-d)}{x(y-c)(y-d)}\right|}{(y-x)^2}\, dy\, dx$$
which is
$$2\left[\mathcal{L}\left(\tfrac{d-1}{d}\right) - \mathcal{L}\left(\tfrac{c-1}{c}\right) + 2\mathcal{L}\left(\tfrac{c-1}{d-1}\right) - 2\mathcal{L}\left(\tfrac{c}{c-d}\right)\right].$$





$$\int_0^1 \int_c^d \frac{\ln\left|\frac{y(x-c)(x-d)}{x(y-c)(y-d)}\right|}{(y-x)^2}\,dy\,dx$$



$$\int_s^0 \int_1^\infty \frac{\ln\left|\frac{y(y-s)(x-1)}{x(x-s)(y-1)}\right|}{(y-x)^2}\,dy\,dx$$

(Bridgeman)

Thank you