Involutions on Higgs bundle moduli spaces

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X — compact Riemann surface

G — complex semisimple Lie group.

G – Higgs bundle (E, φ)

$$\begin{array}{l} E \quad \text{hol. } G\text{-bdle} \\ \downarrow \\ X \end{array}$$

$$\varphi \in H^0(X, \underbrace{E(\mathfrak{g})}_{\substack{\| \\ E \times_{Ad} \mathfrak{g}}} \otimes K) \qquad \sim \text{adj. bdle}$$

(Semi-) stability of (E, φ)

- $\forall$ $P \subsetneq G$ parab. sub.
- $\forall$ $\chi: P \to \mathbb{C}^*$ anti-dom. char.
- $\forall$ reduction of structure group E_P s.t. $\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K)$

$$\deg E_P(\chi) \geq 0$$

Polystability

- (E,φ) semistable
- $\forall P$, reduction to P and χ s.t. $\deg E_P(\chi)=0$, there is a red. to Levi subgroup $L\subset P$ s.t.

$$\varphi \in H^0(X, \underbrace{E_L(L)}_{\mathrm{Lie}(L)} \otimes K)$$

and (E_L,φ) is stable

$\mathcal{M}(G)$ — moduli space of isom. classes of polystable G-Higgs bdles.

- Normal quasi-proj. variety of $\dim = 2(g-1)\dim G$
- (E,φ) simple $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Aut}(E,\varphi) = Z(G)$
- (E,φ) smooth point $\Longleftrightarrow$ (E,φ) stable & simple

$\underset{\text{smooth}}{\mathcal{M}^s(G)}$ — hyperkähler manifold.

– Related to character variety or moduli space of reps. of the fundamental group $\pi_1(X)$

Moduli space of representations

Representation

$$\rho : \pi_1(X) \longrightarrow G$$

ρ reductive $\Longleftrightarrow$ $Ad \circ \rho$ compl. reducible

$$Ad : G \to GL(\mathfrak{g})$$

$$\mathcal{R}(G) = \mathrm{Hom}\,(\pi_1(X), G)^{red} / G$$

Complex alg. variety of dim $= 2(g-1)\dim \mathfrak{g}$

- Theorem

$$\mathcal{M}(G) \underset{\text{homeo}}{\cong} \mathcal{R}(G)$$

Proof is combination of two existence theorems for non-linear PDE's

- Existence of a metric on E (red to maximal cpt) satisfying Hitchin equations (Hitchin '87, Simpson '88)
- Existence of a harmonic metric on a flat G-bdle (Donaldson '87, Corlette '88)

Hyperkähler structure

$\mathcal{M}(G)$ — Cpx structure J_1 coming from X

$\mathcal{R}(G)$ — cpx structure J_2 coming from G

Hyperkähler quotient (Hitchin '87)

Cpx structures J_1, J_2, $J_3 := J_1 J_2$

Metric g on $\mathcal{M}(G)$ by solving Hitchin eqns

Symplectic structures $\omega_i(\cdot,\cdot) = g(J_i\cdot,\cdot)$

Hol. symplectic structures

$\Omega_1 = \omega_2 + i\,\omega_3$, etc.

Interested in:

- Hyperkähler submanifolds
- (J_i, Ω_i) — Lagrangian submanifolds.

Obtained by studying involutions.

Relevant in study of Mirror symmetry & Langlands duality of Higgs bdls.

J_1-holomorphic involutions on $\mathcal{M}(G)$

- $\mathrm{Aut}(G)$ acts on $\mathcal{M}(G)$

$\theta \in \mathrm{Aut}(G)$, $(E,\varphi) \in \mathcal{M}(G)$

$$\theta(E) := E \times_\theta G$$

$$\theta(\varphi) \in \theta(E)(\mathfrak{g}) \otimes K$$

- If $\theta \in \mathrm{Int}(G) \Rightarrow (\theta(E), \theta(\varphi)) \cong (E,\varphi)$

$\Rightarrow \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ acts on $\mathcal{M}(G)$.

Let $a \in \mathrm{Out}_2(G)$

Consider the involutions

$$\iota(a,\pm) : \mathcal{M}(G) \longrightarrow \mathcal{M}(G)$$

$$(E,\varphi) \longmapsto a\cdot(E, \pm\varphi)$$

- To describe the fixed points let

$\theta \in \mathrm{Aut}_2(G)$

$G^\theta = \{ g \in G : \theta(g) = g \}$ — reductive group

$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$

(+1) (−1) eigenspaces of θ

$\mathfrak{g}_+ = \mathrm{Lie}(G^\theta)$

The adjoint rep. of G restricted to G^θ defines reps:

$G^\theta \longrightarrow GL(\mathfrak{g}_\pm)$ + — adjoint rep of G^θ
− — isotropy rep

- $(G, \theta, \pm)$ — Higgs bdles

(E, φ)
E — G^θ — bdle
$\varphi \in H^0(E(\mathfrak{g}_\pm) \otimes K)$

Stability

$\mathcal{M}(G, \theta, \pm)$ — moduli space of semistable $(G, \theta, \pm)$ — Higgs bdles.

- there are maps:

$$\mathcal{M}(G,\theta,\pm) \longrightarrow \mathcal{M}(G)$$
$$(E,\varphi) \longrightarrow (E_G,\tilde{\varphi})$$

$$\left(E_G(\mathfrak{g}) = E(\mathfrak{g}_+) \oplus E(\mathfrak{g}_-)\right)$$

extend φ by zero where appropriate

- $\theta, \theta' \in \mathrm{Aut}_2(G)$

$$\theta \sim \theta' \iff \theta' = f\theta f^{-1}$$
$$f \in \mathrm{Int}(G)$$

there is a map

$$\underbrace{Cl : \mathrm{Aut}_2(G)/\sim}_{\text{finite set}} \longrightarrow \mathrm{Out}_2(G)$$

<u>Theorem</u> (G-Ramanan)

- $$\bigcup_{[\theta]\in Cl^{-1}(a)} \mathrm{im}\left(\mathcal{M}(G,\theta,\pm) \to \mathcal{M}(G)\right) \subset \mathcal{M}^{\iota(a,\pm)}$$
$$\supset \mathcal{M}^{\iota(a,\pm)}_{smooth}$$

- <u>Relation with reps of $\pi_1(X)$</u>

- $\mathcal{M}(G, \theta, +) = \mathcal{M}(G^\theta) \underset{\text{homeo}}{\cong} \mathcal{R}(G^\theta)$

(B, B, B) — branes (<u>hyperkähler subvariety</u>)

- <u>To interpret $\mathcal{M}(G, \theta, -)$</u>, recall

$\sigma : G \to G$ <u>conjugation</u>

$\sigma^2 = 1$

σ antiholomorphic.

$G^\sigma \subset G$ <u>real form</u> of G.

<u>Cartan</u>: $\exists$ conj $\tau \in \mathrm{Conj}(G)$ defining a <u>compact real form</u> such that in each class in $\mathrm{Aut}_2(G)/\sim$ there is a representative θ commuting with τ

$\Rightarrow \sigma = \theta\tau$ conjugation.

<u>Theorem</u>

$\mathcal{M}(G, \theta, -) \underset{\text{homeo}}{\cong} \mathcal{R}(G^\sigma) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(X), G^\sigma)/G^\sigma$

(B, A, A) — branes. ↳ Corlette + Bradlow-G-Gothen-Mundet

• EXAMPLES: See slides

Involutions defined by the centre of G

• $Z := Z(G)$

$H^1(X, Z)$ — group of isomorphism classes of Z-bundles on X

E — G-bdle on X

$\alpha \in H^1(X, Z)$

Have operation

$m : Z \times G \longrightarrow G$ multiplication

$$E \otimes \alpha := \underbrace{(E \times_X \alpha)}_{(Z \times G)\text{-bdle}} \times_m G \quad \text{— } G\text{-bdle}$$

Clearly $E(\mathfrak{g}) = (E \otimes \alpha)(\mathfrak{g})$

• Action of $H^1(X, Z)$ on $\mathcal{M}(G)$

$$\alpha \cdot (E, \varphi) := (E \otimes \alpha, \varphi)$$

- $\mathrm{Out}(G)$ acts on Z and hence on $H^1(X,Z)$

<u>Semi direct product</u>

$$H^1(X,Z) \rtimes \mathrm{Out}(G) \ni (\alpha,a),(\beta,b)$$

$$(\beta,b)\cdot(\alpha,a) = (\beta\otimes b(\alpha), ba)$$

- $H^1(X,Z)\rtimes \mathrm{Out}(G)$ acts on $\mathcal{M}(G)$

$$(\alpha,a)\cdot(E,\varphi) = (a(E)\otimes\alpha, a(\varphi))$$

- To define involutions we consider elements of <u>order 2</u> in $H^1(X,Z)\rtimes\mathrm{Out}(G)$:

$$\Leftarrow (\alpha,a) \text{ s.t } \boxed{\begin{array}{l} a\in \mathrm{Out}_2(G) \\ a(\alpha)=\alpha^{-1}\end{array}}$$

- <u>Consider involutions:</u>

$$\iota(\alpha,a,\pm):\mathcal{M}(G)\longrightarrow\mathcal{M}(G)$$
$$(E,\varphi)\longmapsto(a(E)\otimes\alpha, \pm a(\varphi))$$

Fixed points?

- Consider

$$G_\theta := \{ g \in G : \theta(g) = c_g\, g,\ c_g \in Z \}$$

We have

$$1 \longrightarrow G^\theta \longrightarrow G_\theta \longrightarrow \Gamma_\theta \longrightarrow 1$$

The map

$$\begin{aligned} c : G_\theta &\longrightarrow Z \\ g &\longmapsto c_g \end{aligned}$$

is a homomorphism of groups.

$$\operatorname{Ker} c = G^\theta$$

$\Rightarrow$ $\underline{\text{injective homomorphism}}$

$$\Gamma_\theta \longrightarrow Z \qquad (\Rightarrow \Gamma_\theta \text{ finite}).$$

$\Rightarrow$ $\underline{\text{injective homomorphism}}$

$$H^1(X, \Gamma_\theta) \longrightarrow H^1(X, Z)$$

- The restriction of adj rep of G to G_θ gives reps:

$$G_\theta \longrightarrow GL(\mathfrak{g}_\pm)$$

Can extend the theory of $(G, \theta, \pm)$-Higgs bdles to pairs (E, φ)

E — G_θ-bdle

$\varphi \in H^0(X, E(\mathfrak{g}_\pm) \otimes K)$

The map

$$H^1(X, G_\theta) \longrightarrow H^1(X, \Gamma_\theta)$$
$$E \longmapsto \gamma(E)$$

defines an <u>invariant</u> of E

$\gamma(E) = 1 \iff E$ reduces to a G^θ-bdle.

$\mathcal{M}(G, \theta, \pm, \gamma)$ — moduli spaces.

- <u>Fact</u>: $\boxed{G_\theta = N_G(G^\theta)}$

The maps

$$\mathcal{M}(G,\theta,\pm,\gamma) \longrightarrow \mathcal{M}(G)$$

are <u>injective</u>.

$$\left(\mathcal{M}(G,\theta,\pm,1) = \mathcal{M}(G,\theta,\pm)/\Gamma_\theta \right)$$

- Coming back to inj. homomorphis

$$\Gamma_\theta \longrightarrow Z \qquad (*)$$

– <u>θ acts on Γ_θ</u>: (because θ acts on G_θ and leaves G^θ inv.)

$$\theta(\gamma) = \gamma^{-1} \quad \left(\theta(c_g) = c_g^{-1}\right)$$

– <u>θ acts on Z</u>:

Action <u>only</u> depends on $a = d([\theta]) \in Out_2(G)$.

– inj $(*)$ is θ – <u>equivariant</u>

- Let

$$Z_a = \{ z \in Z : a(z) = z^{-1} \}$$

Prop.

1) $\mathrm{Image}\,(\Gamma_\theta \to Z) \subset Z_a \qquad ([\theta] \in cl^{-1}(a))$

2) The map

$$\bigcup_{[\theta] \in cl^{-1}(a)} \Gamma_\theta \longrightarrow Z_a$$

is surjective

(recall $cl : \mathrm{Aut}_2(G)/_\sim \longrightarrow \mathrm{Out}_2(G)$)

- We have

$$\bigcup_{[\theta] \in cl^{-1}(a)} H^1(X, \Gamma_\theta) \xrightarrow{c} H^1(X, Z_a) = \left\{ \begin{array}{l} \alpha \in H^1(X, Z) : \\ a(\alpha) = \alpha^{-1} \end{array} \right\}$$

Theorem (G–Ramanan)

$$\bigcup_{[\theta] \in cl^{-1}(a),\ \gamma \in c^{-1}(\alpha) \cap H^1(X, \Gamma_\theta)} \mathcal{M}(G, \theta, \pm, \gamma) \subset \mathcal{M}(G)^{\iota(\alpha, a, \pm)}$$

$$\supset \mathcal{M}(G)^{\iota(\alpha, a, \pm)}_{\text{smooth}}$$

Relation with reps:

$$G_\sigma = \{ g \in G : \sigma(g) = c_g g ,\ c_g \in Z \}$$

$$1 \longrightarrow G^\sigma \longrightarrow G_\sigma \longrightarrow \Gamma_\sigma = 1$$

$$\Gamma_\sigma = \Gamma_\theta$$

$$H^1(X, \Gamma_\theta) \cong \mathrm{Hom}(\pi_1(X), \Gamma_\theta)$$

$$\rho : \pi_1(X) \longrightarrow G_\sigma$$

invariant of ρ $\quad \gamma(\rho) : \pi_1(X) \longrightarrow \Gamma_\sigma$

$$\mathcal{R}(G, \gamma) = \{ \rho \in \mathcal{R}(G_\sigma) : \gamma(\rho) = \gamma \}$$

$$\mathcal{M}(G, \theta, -, \gamma) \cong \mathcal{R}(G_\sigma, \gamma)$$ (J_1, Ω_1)-Lagrangian (B, A, A)

$$\mathcal{M}(G, \theta, +, \gamma) \cong \mathcal{R}(G_\theta, \gamma)$$ Hyperkähler subvarieties (B, B, B)