Stacky compactifications of minimal resolutions of singularities of type A_k and gauge theory

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Joint work with

F. Sala *(framed sheaves on stacks)*

M. Pedrini, F. Sala & R. Szabo *(instanton counting on ALE spaces via framed sheaves on stacks)*

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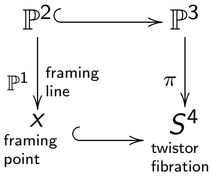
Instantons vs. framed sheaves

Fix the topological type $n \in H^4(S^4, \pi_3(\mathrm{SU}(r)) = \mathbb{Z}$ of $P \xrightarrow{\mathrm{SU}(r)} S^4$
Framed instantons: pairs (∇, Φ)

- ∇ a connection on P such that $*F_\nabla = -F_\nabla$
- $\Phi \in P_{x_0}$

One builds a moduli space of such pairs by considering only gauge transformations that fix the framing

Gauge equiv. classes of framed $SU(r)$-instantons on S^4 are in a one-to-one correspondence with isom. classes of rank r holomorphic bundles on $\mathbb{P}^2$, framed on a "line at infinity" ℓ_∞



Instantons on S^4 correspond to holomorphic bundles on $\mathbb{P}^3$ that are holomorphically trivial on the fibers of π ("twistor lines")

Choose a $\mathbb{P}^2 \subset \mathbb{P}^3$ going thru the twistor line over $x_0 \in S^4$ (which becomes the framing line ℓ_∞)

$\Rightarrow$ get a holomorphic bundle on $\mathbb{P}^2$ framed on ℓ_∞.

$$\mathbb{R}^4 \begin{cases} \text{conformal cpt. } S^4 & \to \quad \text{framed instantons} \\ \text{projective cpt. } \mathbb{P}^2 & \to \quad \text{framed hol. bundles} \end{cases}$$

This may be extended to a correspondence between *ideal framed instantons on* S^4 and *framed torsion-free coherent sheaves on* $\mathbb{P}^2$.

Ideal instantons

For any choice of m points $x_1, \dots, x_m$ in S^4, $(0 \leq m \leq n)$ allow for the degenerations

$$\|F_\nabla\|^2(x_i) = 8\pi^2\delta(x_i)$$

$\Rightarrow$ (Singular) moduli space $\mathcal{M}_0(r, n)$ of ideal framed instantons

$$\begin{array}{ccc}
\mathcal{M}_0^{\text{loc.fr.}}(r,n) & \hookrightarrow & \mathcal{M}(r,n) \\
\Big\| \text{(Donaldson)} & & \Big\downarrow \pi \\
\mathcal{M}_0^{\text{reg}}(r,n) & \hookrightarrow & \mathcal{M}_0(r,n)
\end{array}$$

$\mathcal{M}(r,n) \rightsquigarrow$ Moduli space of torsion-free framed sheaves on $\mathbb{P}^2$

ALE spaces

Finite group Γ acting on $\mathbb{C}^2$ as a subgroup of $SU(2)$

$$X_\Gamma = \widehat{\mathbb{C}^2/\Gamma}$$

with a hyperkähler metric which is approx. Euclidean at infinity

$$\Gamma_k = \left\{ \begin{pmatrix} e^{2\pi i n/k} & 0 \\ 0 & e^{-2\pi i n/k} \end{pmatrix}, \; n = 0, \ldots, k-1 \right\} \simeq \mathbb{Z}_k$$

$$X_k = \widehat{\mathbb{C}^2/\Gamma_k}$$

(ALE space of type A_{k-1})

Stacky compactifications

X_k is a toric variety

$\bar{X}_k = X_k \amalg D_\infty$ normal projective toric compactification of X_k. It is smooth only for $k = 1, 2$, in which cases it coincides with $\mathbb{P}^2$ or $\mathbb{F}_2$.

Want to have this situation

$$\begin{array}{ccc} \mathscr{D}_\infty & \hookrightarrow & \mathscr{X}_k \\ {\scriptstyle r_k}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ D_\infty & \hookrightarrow & \bar{X}_k \end{array}$$

where $\mathscr{X}_k$ is a *smooth proj.* DM stack

$\bar{X}_k$ is a coarse moduli space for $\mathscr{X}_k$

$\mathscr{D}_\infty \xrightarrow{r_k} D_\infty$ is essentially a μ_k-gerbe

π_k is an isomorphism away from $\mathscr{D}_\infty$

Encoding the behavior at infinity

Behavior at infinity of the instantons on X_k

$\updownarrow$

Representations of $\pi_1(S^3/\Gamma) \simeq \Gamma$

$\updownarrow$

Suitable choice of the framing sheaf on $\mathscr{D}_\infty \subset \mathscr{X}_k$

According to a theorem of Bando, if $\bar{X}_k$ is a smooth compactification of X_k obtained by adding a divisor D (with positive normal bundle), bundles on $\bar{X}_k$ framed on D correspond to instantons on X_k with trivial holonomy at infinity

We construct a moduli space of framed sheaves on $\mathscr{X}_k$ containing an open dense subvariety which is a moduli space for instantons on the ALE space with prescribed Chern classes and holonomy at infinity (after work of Eyssidieux and Sala `arXiv:1404.3504`)

Stacks

Stack $[X/G]$: functor $\mathfrak{Sch} \to \mathfrak{Grpds}$

$$S \mapsto \begin{cases} \text{Principal } G\text{-bundles } P \text{ on } S \\ \text{with a } G\text{-equivariant morphism } P \to X \end{cases}$$

Morphisms in this grupoid are the isomorphisms of G-bundles compatible with the morphisms $P \to X$

This generalizes the fact that every scheme defines a functor $\mathfrak{Sch} \to \mathfrak{Sets}$

Examples

- $[*/G](S)$ = principal G-bundles on S
- $[\mathbb{A}^1/\mathbb{G}_m](S)$ = pairs (line bundle L on S, section of L)

Framed sheaves on stacks

$\mathscr{X}$ smooth projective stack with a coarse moduli space $\pi \colon \mathscr{X} \to X$ (normal projective variety)
$D \subset X$ smooth irreducible divisor, $\mathscr{D} \subset \mathscr{X}$ the reduced closed substack with support $\pi^{-1}(D)$ (an effective Cartier divisor)

Fix a locally free sheaf $\mathcal{F}$ on $\mathscr{D}$

Definition

A $(\mathscr{D}, \mathcal{F})$-framed sheaf on $\mathscr{X}$ is a pair $\mathfrak{E} = (\mathcal{E}, \phi)$, where $\mathcal{E}$ is a torsion-free sheaf on $\mathscr{X}$ and $\phi \colon \mathcal{E}|_{\mathscr{D}} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism.

A *morphism of framed sheaves* $f \colon \mathfrak{E} \to \mathfrak{H}$ is a morphism of the underlying coherent sheaves $f \colon \mathcal{E} \to \mathcal{H}$ for which there is an element $\lambda \in k$ such that $\phi_{\mathcal{H}} \circ f = \lambda \phi_{\mathcal{E}}$.

Definition

A *flat family* $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ of framed sheaves on $\mathscr{X}$ parameterized by a scheme S consists of a coherent sheaf $\mathcal{E}$ on $\mathscr{X} \times S$, flat over S, a line bundle $L_{\mathcal{E}}$ on S, and a morphism $\phi_{\mathcal{E}} \colon L_{\mathcal{E}} \to p_{S*}\mathcal{H}om(\mathcal{E}, p_{\mathscr{X}}^{*}(\mathcal{F}))$ which on $\mathcal{D} \times \{s \in S\}$ gives an isomorphism.

Define the **functor of framed sheaves** $\underline{\mathcal{M}}^{\mathscr{X}, \mathscr{D}, \mathcal{F}}$

Assume $\dim \mathscr{X} = 2$ and that the framing sheaf $\mathcal{F}$ is semistable.

Theorem

There is a (fine, quasi-projective) scheme $M^{\mathscr{X}, \mathscr{D}, \mathcal{F}}$ which represents the functor $\underline{\mathcal{M}}^{\mathscr{X}, \mathscr{D}, \mathcal{F}}$

Smoothness of the moduli scheme

Theorem

- *The Zariski tangent space to* $M^{\mathscr{X},\mathscr{D},\mathcal{F}}$ *at a point* $[\mathfrak{E}]$, *with* $\mathfrak{E} = (\mathcal{E}, \phi)$, *is naturally isomorphic to* $\mathbb{E}\mathrm{xt}^1(\mathcal{E}, \mathcal{E} \xrightarrow{\phi} \mathcal{F})$
- *the obstruction to the smoothness of* $M^{\mathscr{X},\mathscr{D},\mathcal{F}}$ *at* $[\mathfrak{E}]$ *lies in the space*

$$\ker\left[\mathbb{E}\mathrm{xt}^2(\mathcal{E}, \mathcal{E} \xrightarrow{\phi} \mathcal{F}) \xrightarrow{\mathrm{tr}} H^2(\mathscr{X}, (\mathcal{O}_{\mathscr{X}}(-\mathscr{D})))\right]$$

If $\mathcal{F} = \mathcal{O}_{\mathscr{D}}^{\oplus r}$

- the tangent space is $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}(-\mathscr{D}))$
- the obstruction lies in the space

$$\ker\left[\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-\mathscr{D})) \xrightarrow{\mathrm{tr}} H^2(\mathscr{X}, (\mathcal{O}_{\mathscr{X}}(-\mathscr{D})))\right]$$

The stack $\mathscr{X}_k$

$\mathscr{X}_k$ will be constructed as a *root stack* over $\bar{X}_k$

What is a root stack?

$\mathscr{X}$ a (separated) Deligne-Mumford stack
$\mathcal{L}$ a line bundle on $\mathscr{X}$
$s \in \Gamma(\mathscr{X}, \mathcal{L})$ and k a positive integer.

The pair $(\mathcal{L}, s)$ defines a morphism $\phi_{\mathcal{L},s} \colon \mathscr{X} \to [\mathbb{A}^1/\mathbb{G}_m]$

Definition

The k-th root stack $\sqrt[k]{(\mathcal{L}, s)/\mathscr{X}}$ is the DM stack defined by the fiber product

$$\begin{CD} \sqrt[k]{(\mathcal{L}, s)/\mathscr{X}} @>>> [\mathbb{A}^1/\mathbb{G}_m] \\ @V{\pi}VV @VV{\theta_k}V \\ \mathscr{X} @>{\phi_{\mathcal{L},s}}>> [\mathbb{A}^1/\mathbb{G}_m] \end{CD}$$

Definition

$\mathscr{X}$ smooth separated DM stack

$\mathscr{D} \subset \mathscr{X}$ effective Cartier divisor in $\mathscr{X}$, k a positive integer.

$\sqrt[k]{\mathscr{D}/\mathscr{X}}$ is the root stack $\sqrt[k]{(\mathcal{O}_{\mathscr{X}}(\mathscr{D}), s_{\mathscr{D}})/\mathscr{X}}$

If s has no zeroes then $\sqrt[k]{(\mathcal{L}, s)/\mathscr{X}} \simeq \mathscr{X}$. So in our case the "stacky" structure of $\sqrt[k]{(\mathcal{L}, s)/\mathscr{X}}$ is concentrated on the pre-image of the framing divisor.

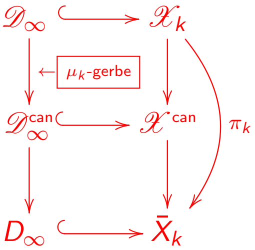
$\mathcal{O}_{\sqrt[k]{\mathscr{D}/\mathscr{X}}}(\tilde{\mathscr{D}})^{\otimes k} \simeq \pi^*(\mathcal{O}_{\mathscr{X}}(\mathscr{D}))$

$\tilde{\mathscr{D}} \to \mathscr{D}$ is a μ_k-gerbe

Line bundles on $\sqrt[k]{\mathscr{D}/\mathscr{X}}$ have the form $\tilde{\mathcal{L}} = \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathscr{X}}(i\tilde{\mathcal{D}})$, $i = 0, \dots, k-1$, for $\mathcal{L}$ a line bundle on X

X_k is a toric variety

$\bar{X}_k = X_k \amalg D_\infty$ normal projective toric compactification of X_k



$$0 \to \mathrm{Cl}(\bar{X}_k) \xrightarrow{\pi_k^*} \mathrm{Pic}(\mathscr{X}_k) \to \mathbb{Z}_k \to 0$$

Picard group

$D_1, \dots, D_{k-1}$ toric divisors in X_k

$$D_i \cdot D_j = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix} = -C_{ij}$$

$$\left(C^{-1}\right)^{ij} = \min(i,j) - \frac{ij}{k} \; .$$

Define the class $\omega_i := -\sum_{j=1}^{k-1} (C^{-1})^{ij} \mathscr{D}_j$ for $i = 1, \dots, k-1$.

Proposition

The Picard group $\mathrm{Pic}(\mathscr{X}_k)$ *of* $\mathscr{X}_k$ *is freely generated over* $\mathbb{Z}$ *by* ω_i *for* $i = 1, \dots, k-1$ *and* $\mathscr{D}_\infty$.

Nekrasov partition function

(the mother of all partition functions)

The Nekrasov partition function is the generating function of the integrals of the equivariant fundamental classes of the moduli spaces of framed sheaves on $\mathbb{P}^2$
It is evaluated by using localization with respect to the toric action

$$Z^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \vec{a}; q) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \mathbf{1},$$

ε_i: parameters in the Lie algebra of $\mathbb{C}^* \times \mathbb{C}^*$

$\vec{a} = (a_1, \ldots, a_r)$: parameters in the Lie algebra of $(\mathbb{C}^*)^r$

$\mathbf{1}$ equivariant fundamental class of $M(r, n)$

$T_\mu = \mathbb{C}^*$, $H^*_{T_\mu}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}[\mu]$
G a T-equivariant locally free sheaf of rank n on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}^{0,\vec{w}}_\infty)$

$$\mathrm{E}_\mu(\mathbf{G}) := \mu^n + c_1(\mathbf{G})_T\, \mu^{n-1} + \cdots + c_n(\mathbf{G})_T \in H^*_{T\times T_\mu}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}^{0,\vec{w}}_\infty)\,;\,\mathbb{Q}\big)\ .$$

$\vec{v} \in \frac{1}{k}\,\mathbb{Z}^{k-1}$ be such that $k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \mod k$ for fixed rank r and holonomy at infinity $\vec{w}$.

$$\mathcal{Z}^*_{\vec{v}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big) = \sum_{\Delta \in \frac{1}{2r\,k}\mathbb{Z}} \mathsf{q}^{\Delta + \frac{1}{2r}\,\vec{v}\cdot C\vec{v}} \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}^{0,\vec{w}}_\infty)} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}^{0,\vec{w}}_\infty)\big)$$
$$\cdot\ \exp\Big(\sum_{s=0}^{\infty}\Big(\sum_{i=1}^{k-1} t_s^{(i)}\,\big[\mathrm{ch}(\mathcal{E})_T/[\mathscr{D}_i]\big]_s + \tau_s\,\big[\mathrm{ch}(\mathcal{E})_T/[X_k]\big]_{s-1}\Big)\Big)\ .$$

(fixed rank, hololonomy at infinity and c_1, sum over Δ)

$$\vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)} = 0 \rightsquigarrow p = 0 \; ; \qquad \vec{\tau} = 0 \rightsquigarrow p = 2$$

$$\mathrm{ch}_T(\tilde{\mathcal{E}})/[X_k] := \sum_{i=1}^{k} \frac{1}{\mathrm{Euler}(T_{p_i} X_k)} \imath^*_{\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}^{0,\vec{w}}_\infty)} \mathrm{ch}_T(\tilde{\mathcal{E}})$$

The *generating function for correlators of p-observables* of $\mathcal{N} = 2$ gauge theory on X_k with an adjoint hypermultiplet of mass μ (deformed partition function) is

$$\mathcal{Z}^*_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big)$$
$$:= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \bmod k}} \vec{\xi}^{\,\vec{v}}\, \mathcal{Z}^*_{\vec{v}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big) \, .$$

(fixed rank and holonomy at infinity, sum over c_1 and Δ

Factorization of the instanton partition function

Full instanton partition function

$$\mathcal{Z}_{X_k}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,) := \sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}}\,\mathcal{Z}_{\vec{v}}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q})\,,$$

(instanton partition functions are obtained by setting $\vec{\tau}=\vec{t}=0$)

$\vec{\xi}^{\,\vec{v}} := \prod_{i=1}^{k-1}\,\xi_i^{v_i}$.

Factorization formula

$$\mathcal{Z}^{*,\mathrm{inst}}_{X_k}\left(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\right)=$$

$$\sum_{\substack{\vec{v}\in\frac{1}{k}\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{\mathbf{v}}} \mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha} \prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\frac{\ell^{(n)}_{\vec{v}_{\beta\alpha}}\left(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}+\mu\right)}{\ell^{(n)}_{\vec{v}_{\beta\alpha}}\left(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\right)}$$

$$\times\prod_{i=1}^{k}\mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}\left(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)},\mu;\mathsf{q}\right),$$

where $\vec{\mathbf{v}}=(\vec{v}_1,\ldots,\vec{v}_r)$, $\vec{v}_{\beta\alpha}=\vec{v}_\beta-\vec{v}_\alpha$ and $a_{\beta\alpha}=a_\beta-a_\alpha$
$\mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}$ is the Nekrasov partition function for the $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ with gauge group $U(r)$.
$\ell^{(n)}_{\vec{v}_{\beta\alpha}}$ are the *edge contributions*

Comparison with existing results

- For $k = 2$ the relevant ALE space coincides with the "open" part of the Hirzebruch surface $\mathbb{F}_2$. Our results reproduce those of B., Poghossian and Tanzini, and Gasparim and Liu, op. cit.
- So for $k = 2$ we also have agreement with Bonelli, Maruyoshi, Tanzini, Yagi JHEP 1301 (2013) p. 014, (arXiv:1208.0790)
- Explicit calculations to compare with BMTY have been done for $k = 3$ and $r = 2$; this requires studying a number of cases depending on the values of $\vec{v}$ and $\vec{w}$. Compatibility is obtained in (almost) all cases.

Seiberg-Witten prepotential

The Seiberg-Witten prepotential can be recovered from the partition function for the Ω-deformed $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$ in the limit $\varepsilon_1, \varepsilon_2 \to 0$ (Nekrasov, Nakajima-Yoshioka, Nekrasov-Okounkov).

We prove analogous results for gauge theory on X_k.
Set $\tilde{k} = k/2$ for even k and $\tilde{k} = k$ for odd k.

Theorem

$F^{*,\mathrm{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,) := -\tilde{k}\,\varepsilon_1\,\varepsilon_2 \log \mathcal{Z}^{*,\mathrm{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$ and

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} F^{*,\mathrm{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,) = \boxed{\frac{1}{k}}\,\mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a}, \mu; \mathsf{q}) \;,$$

where $\mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a}, \mu; \mathsf{q})$ is the instanton part of the Seiberg-Witten prepotential of $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$.

Blowup formula for $k = 2$

$\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, t)$ deformed partition function
$\mathcal{Z}^{*}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, \vec{\tau}, \vec{t}\,)$ specialized at $\vec{\tau} = \vec{0}$ and $\vec{t} = (0, -t, 0, \dots)$.
$\Theta\left[\begin{smallmatrix}\vec{\mu}\\ \vec{\nu}\end{smallmatrix}\right](\vec{\zeta}\,|\,\tau)$ Riemann theta-function with characteristic $\left[\begin{smallmatrix}\vec{\mu}\\ \vec{\nu}\end{smallmatrix}\right]$ on the Seiberg-Witten curve Σ for $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$.

Theorem

$\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \xi, t) / \mathcal{Z}^{*,\mathrm{inst}}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \xi)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$, and

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \xi, t)}{\mathcal{Z}^{*,\mathrm{inst}}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \xi)} = \exp\left(\left(\mathsf{q}\,\frac{\partial}{\partial \mathsf{q}}\right)^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a}, \mu; \mathsf{q})\, t^2 + 2\pi\,\mathrm{i} \sum_{\alpha = w_0 + 1}^{r} \zeta_\alpha\right) \frac{\Theta\left[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\right](C\,(\vec{\zeta} + \vec{\kappa})\,|\,C\,\tau)}{\Theta\left[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\right](C\,\vec{\kappa}\,|\,C\,\tau)},$$

where

$$\zeta_\alpha := -\frac{t}{2\pi\,\mathrm{i}}\left(a_\alpha + \mathrm{q}\,\frac{\partial^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}}{\partial \mathrm{q}\,\partial a_\alpha}(\vec{a},\mu;\mathrm{q})\right) ,$$

while $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log(\xi)$ for $\alpha = 1,\dots,r$ and

$$\nu_\alpha := \begin{cases} \displaystyle\sum_{\beta=w_0+1}^{r} \log\left((a_\beta - a_\alpha)^2 - \mu^2\right) - \frac{2\pi\,\mathrm{i}\,w_1}{r}\,\tau_0 \\ \displaystyle\qquad + \sum_{\beta=w_0+1}^{r} \frac{\partial^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\mu;\mathrm{q}) & \text{for } \alpha = 1,\dots,w_0 \ , \\ \displaystyle -\sum_{\beta=1}^{w_0} \log\left((a_\beta - a_\alpha)^2 - \mu^2\right) + \frac{4\pi\,\mathrm{i}\,w_0}{r}\,\tau_0 & \text{for } \alpha = w_0+1,\dots,r \ . \end{cases}$$

The blowup equation underlies the modularity properties of the partition function and correlators of quadratic 2-observables on the Seiberg-Witten curve for $\mathcal{N} = 2^*$ gauge theory on X_2 with period matrix τ twisted by the A_1 Cartan matrix C, and it generalizes the representation of the Vafa-Witten partition function at $\mu = 0$ in terms of modular forms.

If the fixed holonomy at infinity is trivial, i.e., $\vec{w} = (w_0, w_1) = (r, 0)$, the characteristic vector $\vec{\nu} \in \mathbb{C}^r$ vanishes and our result is in the line of Nakajima-Yoshioka's (2005).

In general, the nontrivial holonomy at infinity is encoded in $\vec{\nu}$.

Comments and perspectives

- First rigorous definition of partition functions for ALE spaces
- We have obtained "blowup" formulas, unlike previous attempts (e.g., by Fucito, Morales, Poghossian)
- Are "our" moduli spaces quiver varieties à la Nakajima? (true for $r = 1$ due to work by Kuznetsov)
- For $\mu \to 0$ we obtain the Vafa-Witten partition function for $\mathcal{N} = 4$ SYM on ALE spaces; we get the same result as Fuji-Minabe (they computed the Euler characteristic of the moduli space of $\mathbb{Z}_k$-invariant framed sheaves on $\mathbb{P}^2$) $\Rightarrow$ compute the Poincaré polynomials of our moduli spaces — done for $k = 1$ ($\mathbb{P}^2$, Nakajima — no stacky structure) and $k = 2$ ($\mathbb{F}_2$, Bruzzo-Poghossian-Tanzini) — stacky structure already present)