The gradient heat flow of Higgs pairs

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Abstract: In this talk, I'll introduce our recent work (joint with Li, Jiayu) on the limiting behavior of the gradient heat flow of Higgs pairs.

Over general Kähler manifolds, we prove that:

- The isomorphism class of the limiting Higgs sheaf is given by the double dual of the graded Higgs sheaves associate to Harder-Narasimhan-Seshadri filtration of the initial Higgs bundle.

Higgs pairs

Given a complex vector bundle E over a compact Kähler manifold (M, ω), suppose that there is a Hermitian structure H_0 on the bundle E. Let $\mathcal{A}_{H_0}$ denote the space of connections of E compatible with metric H_0, and $\mathcal{A}_{H_0}^{1,1}$ denote the space of unitary integrable connections of E

- A pair $(A, \phi) \in \mathcal{A}_{H_0}^{1,1} \times \Omega^{1,0}(End(E))$ is called a **Higgs pair** if the relations $\overline{\partial}_A \phi = 0$ and $\phi \wedge \phi = 0$ are satisfied.

Higgs bundles

A holomorphic bundle $(E, \overline{\partial}_E)$ coupled with one Higgs field $\phi \in \Omega^{1,0}(End(E))$ which satisfying $\overline{\partial}_E \phi = 0$ and $\phi \wedge \phi = 0$ will be called by a **Higgs bundle** $(E, \overline{\partial}_E)$.

- One Higgs pair (A, ϕ) on complex vector bundle E will determine one Higgs structure on E, i.e. one Higgs bundle $(E, \overline{\partial}_A, \phi)$.

The slope of subsheaf

On a Kähler manifold (M, ω), the ω-slope $\mu(F)$ of a torsion-free coherent analytic sheaf F is defined by:

$$\mu_\omega(F) = \frac{deg_\omega(F)}{rank(F)} = \frac{1}{rank(F)} \int_M C_1(F) \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{1}$$

The stability of Higgs sheaf

A torsion-free Higgs sheaf (V, A, ϕ) is said ω-**stable** (resp.ω-semistable) if the usual stability condition $\mu_\omega(F) < \mu_\omega(V)$ $(\leq)$ hold for all proper ϕ-invariant sub-sheaves.

- When the Kähler form is understood we shall sometimes refer to (V, A, ϕ) simply stable or semistable.

H-E metrics on Higgs bundles

A Hermitian metric H in Higgs bundle (E, ϕ) is called **Hermitian-Einstein** (or Hermitian Yang-Mills) if the curvature F of the (in general non-integrable and non-unitary) connection $D_{H,\phi} = D_H + \phi + \phi^{*H}$ (be called by Hitchin-Simpson connection) satisfies the Einstein condition, i.e

$$\sqrt{-1}\Lambda_\omega(F_H + [\phi, \phi^{*H}]) = \lambda Id_E, \tag{2}$$

where D_H is the Chern connection, ϕ^{*H} is the adjoint of ϕ with respect to the metric H.

Higgs bundles (continued)

- Higgs bundles first emerged twenty years ago in Hitchin's (1987) study of the self-dual equations on Riemann surfaces and Simpson's (1988) work on nonabelian Hodge theory.
- Hitchin and Simpson proved that: **a Higgs bundle admits a Hermitian-Einstein metric iff it's Higgs poly-stable**. This is a Higgs bundle version of classical Hitchin-Kobayashi correspondence or the Donaldson-Uhlenbeck-Yau theorem.

Approximate Hermitian-Einstein structure

We say a Higgs bundle (E, ϕ) admits an **approximate Hermitian-Einstein structure** if for every positive ϵ, there is a Hermitian metric H such that that

$$\max_M |\sqrt{-1}\Lambda_\omega(F_H + [\phi, \phi^{*H}]) - \lambda Id|_H < \epsilon. \tag{3}$$

Higgs bundle

- Bruzzo and Otero (2007) prove: for Higgs bundle, " approximate Hermitian-Einstein structure" $\Longrightarrow$ " semi-stability".
- JiaYu Li and Zhang (2012) prove for Higgs bundle general Kähler manifold, "semi-stability" $\Longrightarrow$ " approximate Hermitian-Einstein structure".

A Bogomolov type inequality

As an application, we get a Bogomolov type inequality for semi-stable Higgs bundle, i.e. If $(E, \overline{\partial}_E, \phi)$ is Higgs semi-stable, then we have

$$4\pi^2 \int_M (2C_2(E) - \frac{r-1}{r} C_1(E) \wedge C_1(E)) \frac{\omega^{n-2}}{(n-2)!} \geq 0. \qquad (4)$$

The Harder-Narasimhan filtration

Let $(E, A, \phi) \to (M, \omega)$ be a Higgs bundle (not semi-stable). Then there is a filtration of E by ϕ-invariant coherent sub-sheaves

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E,$$

called the Harder-Narasimhan filtration of Higgs bundle (E, A, ϕ) (abbr, HN-filtration), such that $Q_i = E_i/E_{i-1}$ is torsion-free and Higgs semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $Gr^{hn}(E, A, \phi) = \oplus_{i=1}^{l} Q_i$ is uniquely determined by the isomorphism class of (E, A, ϕ).

The Seshadri filtration

Let (V, ϕ) be a semistable Higgs sheaf over a Kähler manifold (M, ω), then there is a filtration of V by ϕ-invariant subsheaf

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V,$$

called the Seshadri filtration of (V, ϕ), such that V_i/V_{i-1} is torsion-free and Higgs stable. Moreover, $\mu(V_i/V_{i-1}) = \mu(V)$ for each i, and the associated graded object $Gr^s(V, \phi) = \oplus_{i=1}^{l} V_i/V_{i-1}$ is uniquely determined by the isomorphism class of (V, ϕ).

The Harder-Narasimhan-Seshadri filtration

Let (E, A, ϕ) be a Higgs bundle over a Kähler manifold (M, ω). Then there is a double filtration , called a Harder-Narasimhan-Seshadri filtration of Higgs bundle (E, A, ϕ) (abbr, HNS-filtration), with the following properties: if $\{E_i\}_{i=1}^{l}$ is the HN filtration of (E, A, ϕ), then

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i$$

and the successive quotient $Q_{i,j} = E_{i,j}/E_{i,j-1}$ are Higgs stable torsion-free sheaves. Moreover, $\mu(Q_{i,j}) = \mu(Q_{i,j+1})$ and $\mu(Q_{i,j}) > \mu(Q_{i+1,j})$, the **associated graded object**:

$$Gr^{hns}(E, A, \phi) = \oplus_{i=1}^{l} \oplus_{j=1}^{l_i} Q_{i,j}$$

is uniquely determined by the isomorphism class of (E, A, ϕ).

The Harder-Narasimhan type

For a Higgs bundle (E, A, ϕ) of rank R, construct a nonincreasing R-tuple of numbers

$$\vec{\mu}(E, A, \phi) = (\mu_1, \cdots, \mu_R) \tag{5}$$

from the HN filtration by setting: $\mu_i = \mu(Q_j)$, for $rk(E_{j-1}) + 1 \leq i \leq rk(E_j)$.

- We call $\vec{\mu}(E, A, \phi)$ the Harder-Narasimhan type of (E, A, ϕ).

Yang-Mills equation

The **Yang-Mills functional** is defined on $\mathcal{A}_{H_0}$ by

$$YM(A) = \int_M |F_A|^2 \, dV_\omega, \tag{6}$$

where dV_ω is the volume form of ω. We call A a **Yang-Mills connection** of E if A is a critical point of the Yang-Mills functional i.e. it satisfies the Yang-Mills equation

$$D_A^* F_A = 0, \tag{7}$$

where D_A^* is the adjoint operator of the covariant differentiation associated with the connection D_A.

Yang-Mills Flow

The Yang-Mills flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A, \tag{8}$$

was first introduced by Atiyah-Bott in [AB].

Donaldson [Do] used it to establish the connection between Hermitian-Yang-Mills connections and stable holomorphic bundles.

- Donaldson proved the global existence of the Yang-Mills flow in a holomorphic bundle over a projective manifold, and proved the convergence of the flow at infinity in the case that the holomorphic bundle is stable.

Atiyah-Bott's conjecture

In general case.

Atiyah and Bott [AB] first point out that there is a relation between the limiting of the Yang-Mills flow and the Harder-Narasimhan filtration of the initial holomorphic stucture over Riemann surface. Bando and Siu [BS] conjectured it will be true for higher dimension.

Atiyah-Bott's conjecture

- Daskalopoulos [Da] (1992) proved the above conjecture over Riemann surfaces.
- Daskalopoulos and Wentworth [DW] (2004) for Kähler surfaces.

- In [HT] (2004), Hong and Tian considered the bubbling phenomena of Yang-Mills flow on general Riemannian manifolds.
- Higher dimensional case by Jacob (2011) and Sibley (2012) independently.

The Yang-Mills-Higgs functional of Higgs pairs

Let $\mathcal{B}_{(E,H_0)}$ denote the space of all Higgs pairs on Hermitian vector bundle (E, H_0). The Yang-Mills-Higgs functional on $\mathcal{B}_{(E,H_0)}$ is defined by:

$$YMH(A,\phi) = \int_M (|F_A + [\phi,\phi^*]|^2 + 2|\partial_A\phi|^2)\, dV_g .$$

It's Euler-Lagrange equation:

$$\left\{ \begin{array}{l} D_A^* F_A + \sqrt{-1}(\partial_A \Lambda_\omega - \overline{\partial}_A \Lambda_\omega)[\phi,\phi^*] = 0, \\ [\sqrt{-1}\Lambda_\omega(F_A + [\phi,\phi^*]),\phi] = 0, \end{array} \right. \tag{9}$$

where the operator Λ_ω is the contraction with ω, and ϕ^* denotes the dual of ϕ respect to the given metric H_0.

By Chern-Weil theory, it is easy to check that if (A, ϕ) satisfies the following **Hermitian-Einstein** equation

$$\sqrt{-1}\Lambda_\omega(F_A + [\phi, \phi^*]) = \lambda Id_E, \tag{10}$$

then it must satisfy the above Euler-Lagrange equation. In fact, it is the absolute **minima** of the above Yang-Mills-Higgs functional.

The gradient flow of Higgs pairs

Now, we want to study the gradient flow of the Yang-Mills-Higgs functional of Higgs pairs, i.e.

$$\begin{cases} \frac{\partial A}{\partial t} = -D_A^* F_A - \sqrt{-1}(\partial_A \Lambda_\omega - \overline{\partial}_A \Lambda_\omega)[\phi, \phi^*], \\ \frac{\partial \phi}{\partial t} = -[\sqrt{-1}\Lambda_\omega(F_A + [\phi, \phi^*]), \phi], \end{cases} \tag{11}$$

- The above flow was introduced By G. Wilkin firstly (2006), which can be seen as a Higgs pairs version of the Yang-Mills flow.

Atiyah-Bott-Bando-Siu conjecture

It is natural to consider the Higgs pairs version of Atiyah-Bott-Bando-Siu conjecture.

- Riemann surfaces case by Wilkin (2008);
- Kähler surfaces case by Li-Zhang (2011);
- Higher dimensional case by Li-Zhang (2013, preprint).

Theorem

Let (E, H_0) be a Hermitian vector bundle on a compact Kähler manifold (M, ω), and $(A(t), \phi(t))$ be a global smooth solution of the above gradient flow (11) with smooth initial Higgs pair (A_0, ϕ_0).
We prove

- *There is a sequence $\{t_i\}$ such that, as $t_i \to \infty$, $(A, \phi)(x, t_i)$ converges, modulo gauge transformations, to a Yang-Mills Higgs pair (A_∞, ϕ_∞) on Hermitian vector bundle (E_∞, H_∞) in C^∞_{loc} topology outside a closed set $\Sigma^{an} \subset M$, where Σ^{an} is of Hausdorff codimensional at leat* 4. The limiting Higgs pair will be call by a **Uhlenbeck limit**.

Blow-up set

$$\Sigma^{an} = \cap_{0<r<i(M)}\{x \in M : \liminf_{k\to\infty} r^{4-2m} \int_{B_r(x)} e(A,\phi)(\cdot, t_k) dV_g \geq \epsilon\}. \quad (12)$$

Here $e(A,\phi) = |F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2$.

- The limiting $(E_\infty, A_\infty, \phi_\infty)$ can be extended to the whole M as a reflexive Higgs sheaf with a holomorphic orthogonal splitting:

$$(E_\infty, H_\infty, A_\infty, \phi_\infty) = \bigoplus_{i=1}^{l}(E^i_\infty, H^i_\infty, A^i_\infty, \phi^i_\infty), \tag{13}$$

  where H^i_∞ is a Hermitian-Einstein metrics on the the reflexive Higgs sheaf $(E^i_\infty, A^i_\infty, \phi^i_\infty)$.

- The Harder-Narasimhan type of the Higgs sheaf $(E_\infty, A_\infty, \phi_\infty)$ is the same as that of the Higgs bundle (E, A_0, ϕ_0) which determined by the initial Higgs pair (A_0, ϕ_0).
- $$(E_\infty, A_\infty, \phi_\infty) \simeq Gr_\omega^{hns}(E, \overline{\partial}_{A_0}, \phi_0)^{**},$$
  where $Gr_\omega^{hns}(E, A_0, \phi_0)^{**}$ is the double dual of the associated graded object of the Harder-Narasimhan-Seshadri filtration of the initial Higgs bundle (E, A_0, ϕ_0).

Key points on our proof.

Since we proved the Higgs field $\phi(t)$ are uniformly bounded, so we can follow the basic idea in Kähler surface case by Daskalopoulos and Wentworth, but there are two points where we need new argument for higher dimensional case.

- the existence of L^p-approximate critical Hermitian metric;
- to construct a nonzero holomorphic map between two sub-sheaves.

The long-time existence of the gradient flow

Let (A_0, ϕ_0) be an initial Higgs pair on (E, H_0). Then we consider the following heat flow for Hermitian metrics on the Higgs bundle (E, A_0, ϕ_0) with initial metric H_0:

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_H + [\phi_0, \phi_0^{*H}]) - \lambda Id_E), \qquad (14)$$

where F_H is the curvature form of the Chern connection A_H on E with respect to H.

Simpson proved that solutions to the above nonlinear heat equation exist for all time and depend continuously on the initial condition H_0.

Following Donaldson's argument, by solving a **linear ODE**, we can prove that the gradient heat (11) has an unique solution $(A(t), \phi(t))$ in the complex gauge orbit of (A_0, ϕ_0).

In fact,

$$A(t) = g(t)(A_0) \quad \textit{and} \quad \phi(t) = g(t)(\phi_0), \tag{15}$$

where $g(t) \in \mathcal{G}^C$ satisfies $g^{*H_0}(t) \circ g(t) = h(t) = H_0^{-1}H(t)$, and $H(t)$ is the solution of the above Donaldson's heat flow (14) on Higgs bundle (E, A_0, ϕ_0) with initial metric H_0.

Basic estimates

Let (A, ϕ) be a solution of the heat flow (1.6) with initial Higgs pair (A_0, ϕ_0), we have:

- $$YMH(t) + 2\int_0^t \int_M (|\frac{\partial A}{\partial t}|^2 + 2|\frac{\partial \phi}{\partial t}|^2) = YMH(0).$$

For simplicity, we set

$$\theta = \Lambda_\omega(F_A + [\phi, \phi^*]). \tag{16}$$

We have:

- $$(\triangle - \frac{\partial}{\partial t})|\theta|^2 \geq 0. \tag{17}$$

- $$I(t) = \int_M |D_A\theta|^2 + 2|[\theta, \phi]|^2 \to 0, \quad (t \to \infty). \tag{18}$$

estimate on the Higgs field

By direct calculation, we can obtain

$$(\triangle - \frac{\partial}{\partial t})|\phi|^2_{H_0} \geq 2|\partial_A \phi|^2_{H_0} + C_1|\phi|^4_{H_0} - C_2|\phi|^2_{H_0}. \tag{19}$$

By the Maximum principle, we have a uniform bound on $|\phi(t)|_{H_0}$.

Furthermore, we have:

- a monotonicity inequality for $(A(t), \phi(t))$;
- ϵ-regularity estimate.

Following Hong-Tian's (2004) argument, we have:

There exists a sequence $\{t_j\}$ such that, as $t_j \to \infty$, $(A(t_i), \phi(t_i))$ converges, modulo gauge transformations, to a solution (A_∞, ϕ_∞) of the Yang-Mills-Higgs equation (9) on Hermitian vector bundle (E_∞, H_∞) in C^∞_{loc} topology outside $\Sigma^{an} \subset M$, where Σ^{an} is a closed set of Hausdorff codimension 4.

the limiting Higgs pairs

From the Yang-Mills-Higgs equation (9) and the Kähler identity, we have

$$\begin{aligned} D_{A_\infty}\theta_\infty &= 0, \\ [\theta_\infty, \phi_\infty] &= 0, \end{aligned} \tag{20}$$

where $\theta_\infty = \Lambda_\omega(F_{A_\infty} + [\phi_\infty, \phi_\infty^*])$. Since θ_∞ is **parallel** and $(\sqrt{-1}\theta_\infty)^* = \sqrt{-1}\theta_\infty$, we can decompose E_∞ according to the eigenvalues of $\sqrt{-1}\theta_\infty$. We obtain a **holomorphic orthogonal decomposition** $E_\infty = \bigoplus_{i=1}^{l} E_\infty^i$, and $\phi_\infty : E_\infty^i \to E_\infty^i$.
We have the constant eigenvalues vector $\vec{\lambda}_\infty = (\lambda_1, \cdots, \lambda_R)$, where $\lambda_i \geq \lambda_{i+1}$.

the limiting Higgs pairs

Let H^i_∞ be the restrictions of H_∞ to E^i_∞, ϕ^i_∞ be the restriction of ϕ_∞ to E^i, and $A^i_\infty = A|_{E^i}$. Then $(A^i_\infty, \phi^i_\infty)$ is a Higgs pair on (E^i_∞, H^i_∞) and satisfies

$$\sqrt{-1}\Lambda_\omega(F_{A^i_\infty} + [\phi^i_\infty, (\phi^i_\infty)^*]) = \lambda_i Id_{E^i_\infty}. \tag{21}$$

So $(A^i_\infty, \phi^i_\infty)$ is a Hermitian-Einstein Higgs pair on (E^i_∞, H^i_∞), i.e. $(E^i_\infty, H^i_\infty, A^i, \phi^i_\infty)$ is a Hermitian-Einstein Higgs bundle on $M \setminus \Sigma_{an}$.

Extension of the limiting Higgs structure

Since the Yang-Mills-Higgs functional is decreasing along the gradient flow (11), and $\phi(t)$ is uniformly C^0, then we have:

$$\int_{M\setminus\Sigma_{an}} |F_{A_\infty}|^2_{H_\infty}\frac{\omega^n}{n!} \le C < \infty. \tag{22}$$

Recall that the singularity set Σ_{an} is of Hausdorff codimension 4, ϕ_∞ is holomorphic and C^0 bounded, and every metrics H^i_∞ (or the connection A^i_∞) satisfies the Hermitian-Einstein equation (21), By Bando and Siu's **removable sigularity theorem**, every $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ can be extended to the whole M as a reflexive sheaf (which also be denoted by $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ for simplicity), ϕ^i_∞ and H^i_∞ can be smoothly extended over the place where the sheaf $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ is locally free.

The extension $(E_\infty, H_\infty, A_\infty, \phi_\infty)$ has a holomorphic orthogonal splitting as a direct sum:

$$(E_\infty, H_\infty, A_\infty, \phi_\infty) = \bigoplus_{i=1}^{l} (E^i_\infty, H^i_\infty, A^i_\infty, \phi^i_\infty), \qquad (23)$$

where H^i_∞ is a Hermitian-Einstein metrics on the reflexive Higgs sheaf $(E^i_\infty, A^i_\infty, \phi^i_\infty)$. Since Hermitian-Einstein Higgs sheaf must be semi-stable, so the eigenvalue vector $\vec{\lambda}_\infty = (\lambda_1, \cdots, \lambda_R)$ is just the HN type of the reflexive Higgs sheaf $(E_\infty, A_\infty, \phi_\infty)$.

outline of our proof

Now, we only need to prove that:
$(E_\infty, A_\infty, \phi_\infty)$ is **holomorphically isomorphic** to $Gr^{HNS}(E, A_0, \phi_0)^{**}$ in Higgs sheaf sense, i.e. there is a holomorphical isomorphic map $f_\infty : \oplus_{i=1}^{l} Q_i^{**} \to (E_\infty, A_\infty)$ such that $f_\infty(Q_i^{**})$ is ϕ_∞-invariant for all i.

We will prove the result inductively on the length of the HNS filtration.

outline of our proof

- (1$*$) Let $\vec{\mu}_0$ be the HN type of the initial Higgs bundle (E, A_0, ϕ_0), in the following, we firstly prove that $\vec{\lambda}_\infty = \vec{\mu}_0$, i.e. the limiting Higgs sheaf has the same HN type of the initial Higgs bundle.

outline of our proof

- (2) Denote $(A_j, \phi_j) = (A(t_j), \phi(t_j)) = g_j(A_0, \phi_0)$, where g_j be the complex gauge transformation. Set $\Sigma = \Sigma^{an.} \cup \Sigma^{alg.}$ and $\Omega = M \setminus \Sigma$. Let $S = E_1$ be the maximal Higgs stable subsheaf of (E, A_0, ϕ_0), now $S|_\Omega$ is a holomorphic subbundle, and let $i : S|_\Omega \to (E, \overline{\partial}_{A_0})$ be the ϕ_0-invariant holomorphic inclusion. Define the map $f_j : S|_\Omega \to (E, \overline{\partial}_{A_j})$ by $f_j = g_j \circ i$, it is easy to check that

$$\overline{\partial}_{A_0, A_j} f_j = 0, \quad f_j \circ \phi_0 = \phi_j \circ f_j, \tag{24}$$

  i.e. f_j is a ϕ-invariant holomorphic map.

outline of our proof

- (3$*$) We can prove: By choosing a subsequence, up to rescale, f_j converges in $C^{\infty}_{loc}(\Omega)$ to some non-zero ϕ-invariant holomorphic map f_{∞}.
- (4) Let $\pi_1^{(j)}$ denotes the projection to $g_j(S)$. By choosing a subsequence, we can show that $\pi_1^{(j)} \to \pi_1^{\infty}$ weakly in L_1^2. By Uhlenbeck-Yau's result, we know that π_1^{∞} determines a Higgs subsheaf E_1^{∞} of $(E_{\infty}, \overline{\partial}_{A_{\infty}}, \phi_{\infty})$, with $rank(E_1^{\infty}) = rank(S)$ and $\mu(E_1^{\infty}) = \mu(S)$. Since in (1), we have show the same HN type, so E_1^{∞} must be semi-stable Higgs sheaf.

outline of our proof

- (5) Since $\pi_1^{(j)} \circ f_j = f_j$, we see that in the limit $\pi_1^{\infty} \circ f_\infty = f_\infty$. From (3), we obtained a nonzero smooth ϕ-invariant holomorphic map $f_\infty : (S, \overline{\partial}_{A_0} \to (E_1^{\infty}, \overline{\partial}_{A_\infty})$ on Ω. By Hartog's theorem, f_∞ can be extends to a Higgs sheaf homomorphism on M. Since S is Higgs stable, By Kobayashi's result, we f_∞ must be injective, then

$$S \simeq E_1^{\infty} = f_\infty(S) \tag{25}$$

on $M \setminus (\Sigma_{al} \cup \Sigma_{an})$.

outline of our proof

- (6)Write $E_{\infty}=S_{\infty}\oplus Q_{\infty}$, and denote $Q=E/S$. We consider the induced Higgs pairs (A_j^Q,ϕ_j^Q) on Q. We can check that Higgs pairs (A_j^Q,ϕ_j^Q) satisfy the following inductive hypotheses.

Inductive hypotheses: *There is a sequence of Higgs structures (A_j^Q,ϕ_j^Q) on Q such that:*
(1) $(A_j^Q,\phi_j^Q)\to(A_{\infty}^{Q_\infty},\phi_{\infty}^{Q_\infty})$ in C^{∞}_{loc} off $\Sigma_{al}\cup\Sigma_{an}$;
(2) $(A_j^Q,\phi_j^Q)=g_j(A_0^Q,\phi_0^Q)$ for some $g_j\in\mathbf{G}^C(Q)$;
(3) $(Q,\overline{\partial}_{A_0^Q},\phi_0^Q)$ and $(Q_{\infty},\overline{\partial}_{A_{\infty}^{Q_\infty}},\phi_{\infty}^{Q_\infty})$ extended to M as reflexive Higgs sheaves with the same HN type;
(4) $\|\phi_j^Q\|_{C^0}$ and $\|\sqrt{-1}\Lambda_{\omega}(F_{A_j^Q})\|_{L^1(\omega)}$ is bounded uniformly in j.

outline of our proof

- (7) By induction, repeating steps (2) to (6), we have

$$E_\infty = \oplus_{i=1}^{l} Q_\infty^i \simeq Gr^{HNS}(E, \overline{\partial}_{A_0}, \phi_0) = \oplus_{i=1}^{l} \oplus_{j=1}^{r_i} Q_{i,j} \quad (26)$$

  on $M \setminus (\Sigma_{al} \cup \Sigma_{an})$.
- (8) By Siu's uniqueness of reflexive extension, we know that there exists a sheaf isomorphism

$$f : (E_\infty, \overline{\partial}_{A_\infty}, \phi_\infty) \to Gr^{HNS}(E, \overline{\partial}_{A_0}, \phi_0)^{**} \quad (27)$$

  on M. This completes the proof .

In the following, we give proofs of step (1) and (3).

The HN type of the limiting Higgs bundles

Let (A_∞, ϕ_∞) be an Uhlenbeck limit. From the above, we know that the constant eigenvalues vector $\vec{\lambda}_\infty = (\lambda_1, \cdots, \lambda_R)$ of $\sqrt{-1}\theta_\infty$ is just the HN type of the limiting Higgs sheaf $(E_\infty, A_\infty, \phi_\infty)$. Let $\vec{\mu}_0$ be the HN type of the initial Higgs bundle (E, A_0, ϕ_0), in the following, we will prove that

$$\vec{\lambda}_\infty = \vec{\mu}_0. \tag{28}$$

- We follow Daskalopoulos and Wentworth's argument (2004) to prove the above equality, for time reason, I will not give it in details and only give the proof of the key point.

Key point in the proof of step (1)

To prove (28), it is sufficiently to prove the existence of L^p**-approximate critical Hermitian metric** for $1 \leq p \leq \alpha_0$.

Let H be a smooth Hermitian metric on the holomorphic bundle $\mathbf{E} = (E, \overline{\partial}_E)$, and let $F = \{F_i\}_{i=1}^{l}$ be the HN-filtration of Higgs bundle $(\mathbf{E}, \phi)$. Associated to each F_i and the metric H we have the unitary projection π_i^H onto F_i. Defining $\Psi^{HN}(E , \phi, H) = \Sigma_{i=1}^{l} \mu_i (\pi_i^H - \pi_{i-1}^H)$.

Fix $\delta > 0$ and $1 \leq p \leq \infty$. An L^p-δ-approximate critical Hermitian metric on a Higgs bundle $(\mathbf{E}, \phi)$ is a smooth H such that

$$\|\frac{\sqrt{-1}}{2\pi}\Lambda_\omega(F_{A_H} + [\phi, \phi^{*H}]) - \Psi^{HN}(\mathbf{E}, \phi, H)\|_{L^p(\omega)} \leq \delta,$$

where A_H is the Chern connection determined by $(\overline{\partial}_E, H)$.

existence of L^p-approximate critical Hermitian metric

By the singularities theorem of Hironaka, we can resolve the singularities of Σ_{alg} (i.e. where the related sheaves are not locally free) and obtain a filtration by subbundles.

- Let $\{E_{i,j}\}$ be the HNS-filtration of a Higgs bundle (E, A, ϕ) on complex manifold M and let $Q_{i,j} = E_{i,j}/E_{i,j-1}$. Then there is a finite sequence of blowups along complex submanifolds of M whose composition $\pi : \tilde{M} \to M$ enjoys the following properties. There is a filtration $\{\tilde{E}_{i,j}\}$ by $\tilde{\phi}$-subbundles such that $\tilde{E}_{i,j}$ is the saturation of $\pi^* E_{i,j}$, and $\pi_* \tilde{E}_{i,j} = E_{i,j}$ and $Q_{i,j}^{**} = (\pi_* \tilde{Q}_{i,j})^{**}$, where $\tilde{\phi} = \pi^* \phi$.

ω_ϵ-stability

For simplicity, we suppose that there is only one blow-up

$$\pi : \tilde{M} \to M. \tag{29}$$

On $\tilde{M}$, we have a sequence of Kähler metrics

$$\omega_\epsilon = \pi^* \omega + \epsilon \eta, \tag{30}$$

where η is a Kähler metric on $\tilde{M}$.

- We can prove that every pull back Higgs subbundle $\tilde{E}_{i,j}$ is ω_ϵ-stable for all $0 < \epsilon \leq \epsilon_0$.

L^∞-approximate critical Hermitian metric on $\tilde{M}$

- By Simpson's result: every stable Higgs bundle must have a Hermitian-Einstein metric, and following Donaldson's argument, we have:

  For any $\tilde{\delta} > 0$ and any $0 < \epsilon < \epsilon^*$, there is a smooth Hermitian metric $\tilde{H}$ on $\tilde{\mathbf{E}}$ such that

  $$\|\frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_\epsilon}(F_{(\overline{\partial}_{\tilde{E}},\tilde{H})} + [\tilde{\phi}, \tilde{\phi}^{*\tilde{H}}]) - \Psi(\tilde{F}, (\mu_{\epsilon,1}, \cdots, \mu_{\epsilon,l}), \tilde{H})\|_{L^\infty} \leq \tilde{\delta}, (31$$

  where $(\overline{\partial}_{\tilde{E}}, \tilde{H})$ denotes the Chern connection with respect to holomorphic structure $\overline{\partial}_{\tilde{E}}$ and metric $\tilde{H}$, and $\mu_{\epsilon,i}$ is the slope of quotient $\tilde{Q}_i$ with respect to the metric ω_ϵ.

L^p-approximate metric independent of ϵ

Using Sibley's integral estimate (arXiv: 1206.5491, Lemma 5.3.), we have:

- For any $\delta' > 0$ and any $1 \leq p < 1 + \frac{1}{2k-1}$ there are $\epsilon_1 > 0$ and a smooth Hermitian metric $\tilde{H}_1$ on $\tilde{E}$ such that

  $$\|\frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_\epsilon}(F_{(\overline{\partial}_{\tilde{E}},\tilde{H}_1)} + [\tilde{\phi}, \tilde{\phi}^{*\tilde{H}_1}]) - \Psi(\tilde{F}, (\mu_1, \cdots, \mu_l), \tilde{H}_1)\|_{L^p(\tilde{M},\omega_\epsilon} \leq \delta',$$

  for all $0 < \epsilon \leq \epsilon_1$.

Cut-off argument

Let's consider a sequence of open neighborhood U_R of Σ_{alg}. For every R, we choose a smooth cut-off function ψ_R which supported in U_R and identically 1 on $U_{\frac{R}{2}}$, $0 \leq \psi_R \leq 1$, and furthermore $|\partial\psi_R|^2_\omega + |\partial\overline{\partial}\psi_R|_\omega \leq CR^{-2}$, where C is a positive constant independent of R.

- Let H_D be a smooth Hermitian metric on bundle E, and $\tilde{H}_1$ be the metric on $\tilde{E}$ such that (32) hold for all $0 < \epsilon \leq \epsilon_1$ and where $\delta \leq \frac{\delta'}{4}$. Consider that E is isomorphic to $\tilde{E}$ outsides Σ_{alg}, we can define

$$H_R = (1 - \psi_R)\tilde{H}_1 + \psi_R H_D \tag{33}$$

on bundle E.

- After obtaining some uniformly estimates, recall Σ_{alg} has Hausdorff dimension at most $2n - 4$, by choosing R small enough, H_R is the L^p-approximate critical Hermitian which we needed.

Analytic difficulty

Now, we should need to construct **non-zero holomorphic maps** from subsheaves in the HNS filtration of the original Higgs bundle to the limiting reflexive sheaf. Since in our case, we have only L^1 bound on the curvature. This bring some difficulties in analytic aspect.

Assumption

Let (M, ω) be a Kähler manifold, (E, A_0, ϕ_0) be a Higgs sheaf on M with Hermitian metric H_0, S be a Higgs sub-sheaf of (E, A_0, ϕ_0), and $(A_j, \phi_j) = g_j(A_0, \phi_0)$ be a sequence of Higgs pairs on E, where g_j is a sequence of complex gauge transformations.

- Assume that (A_j, ϕ_j) converges to (A_∞, ϕ_∞) outside a closed subset Σ_{An} of Hausdorff complex codimension 2 ,
- $|\sqrt{-1}\Lambda_\omega(F_{A_j})|_{H_0}$ is bounded uniformly in j in $L^1(\omega_0)$.

Further assumption

- Suppose that there exits a sequence of blow-ups: $\pi_i : \overline{M}_i \to \overline{M}_{i-1}$, $i = 1, \cdots, r$ (where $\overline{M}_0 = M$, every π_i is blow up with non-singular center; denoting $\pi = \pi_r \circ \cdots \circ \pi_1$); such that $\pi^* E$ and $\pi^* S$ are bundles, the pulling back geometric objects $\pi^*(A_0, \phi_0)$, $\pi^* g_j$ and $\pi^* H_0$ can be extended smoothly on the whole M_r.

Remark: *Since we can resolve the singularity set Σ_{alg} by blowing up finitely many times with non-singular center, and the pulling back of the HNS filtration is given by sub-bundles. The sheaf and every geometric objects which we considered are induced by the HNS filtration, so their pulling back are all smooth.*

For simplicity, we assume $r = 1$.

Let $i_0 : (S, \bar{\partial}_{A_0}) \to (E, \bar{\partial}_{A_0})$ be the holomorphic inclusion, then there is subsequence of $g_j \circ i_0$, up to rescale, converges to a non-zero holomorphic map $f_\infty : (S, \overline{\partial}_{A_0}) \to (E_\infty, \overline{\partial}_{A_\infty})$ in C^∞_{loc} off $\Sigma_{alg} \cup \Sigma_{An}$, and $f_\infty \circ \phi_0 = \phi_\infty \circ f_\infty$.

Proof:

- Define the map $\tilde{\eta}_j : (\tilde{S}, \bar{\partial}_{A_0}) \to (\tilde{E}, \bar{\partial}_{A_j})$ by $\tilde{\eta}_j = g_j \circ i_0$. It is easy to check that

$$\overline{\partial}_{A_0, A_j} \tilde{\eta}_j = 0, \quad \tilde{\eta}_j \circ \phi_0 = \phi_j \circ \tilde{\eta}_j, \tag{34}$$

  i.e. $\tilde{\eta}_j$ is a ϕ-invariant holomorphic map.

Evolving Hermitian metric

- Let $H_{j,\epsilon}(t)$ and $H^S_\epsilon(t)$ are the solutions of Donaldson's flow on holomorphic bundles $(\tilde{E}, \overline{\partial}_{A_j})$ and $(\tilde{S}, \overline{\partial}_{A_0})$ with the fixed initial metrics $\tilde{H}_0$ and H^S_0 and with respect to the metric ω_ϵ, i.e. it satisfies the following heat equation

$$H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}\Lambda_{\omega_\epsilon} F_H. \qquad (35)$$

- By directly calculation, we have

$$(\triangle_{\epsilon}-\frac{\partial}{\partial t})|\tilde{\eta}_j|^2_{H_{\epsilon}^S(t),H_{j,\epsilon}(t)}\geq 0. \tag{36}$$

- By the Maximum principle, we have: for $t>0$

$$|\tilde{\eta}_j|^2_{H_{\epsilon}^S(t_0+t),H_{j,\epsilon}(t_0+t)}(x)\leq\int_{\tilde{M}}K_{\epsilon}(t,x,y)|\tilde{\eta}_j|^2_{H_{\epsilon}^S(t_0),H_{j,\epsilon}(t_0)}\frac{\omega_{\epsilon}^n}{n!}, \tag{37}$$

and

$$|\sqrt{-1}\Lambda_{\omega_{\epsilon}}(F_{H_{j,\epsilon}(t)})|_{H_{j,\epsilon}(t)}(x)\leq\int_{\tilde{M}}K_{\epsilon}(t,x,y)|\sqrt{-1}\Lambda_{\omega_{\epsilon}}(F_{\tilde{A}_j})|_{\tilde{H}_0}\frac{\omega_{\epsilon}^n}{n!}, \tag{38}$$

where $K_{\epsilon}(t,x,y)$ is the heat kernel of the Laplacian with respect to ω_{ϵ}.

Taking the limit $\epsilon \to 0$, and following Bando-Siu's argument, we have a solution of the heat flow (35) $H_j(t)$ (and $H^S(t)$) on M. And, we also have

$$|\Lambda_\omega(F_{H_j(t)})|_{H_j(t)}(x) \leq \int_M K(t,x,y)|\Lambda_\omega(F_{A_j})|_{H_0}\frac{\omega^n}{n!}, \tag{39}$$

and

$$|\tilde{\eta}_j|^2_{H^S(t_0+t),H_j(t_0+t)}(x) \leq \int_M K(t,x,y)|\tilde{\eta}_j|^2_{H^S(t_0),H_j(t_0)}\frac{\omega^n}{n!}, \tag{40}$$

for all x outside Σ_{alg}, where $K(t,x,y)$ is the heat kernel of (M,ω).

- We can prove that: there exists a unform constant C_F such that

$$e^{-C_F\delta} \leq \frac{|\tilde{\eta}_j|^2_{H^S(t_0+\delta),H_j(t_0+\delta)}}{|\tilde{\eta}_j|^2_{H^S(t_0),H_j(t_0)}}(x) \leq e^{C_F\delta}, \tag{41}$$

- From (40) and (41), we have

$$\begin{aligned} &|\tilde{\eta}_j|^2_{H^S(1),H_j(1)}(x) \\ &\leq C_a \int_M |\tilde{\eta}_j|^2_{H^S(1),H_j(1)} \frac{\omega^n}{n!} \end{aligned} \tag{42}$$

for $x \in M \setminus \Sigma_{alg}$.

C^0-estimate

- Rescaling $\tilde{\eta}_j$, (i.e. setting $f_j = \frac{\tilde{\eta}_j}{\|\tilde{\eta}_j\|_{L^2}}$), we have a sequence ϕ-invariant $\overline{\partial}_{0,j}$-holomorphic map f_j such that

$$|f_j|^2_{H^S(1),H_j(1)} \leq C_a, \quad \int_M |f_j|^2_{H^S(1),H_j(1)} \frac{\omega^n}{n!} = 1. \qquad (43)$$

- We can obtain locally uniform C^0-estimates of metrics $H_j(t)$ and $H^S(t)$, i.e. for any compact subset Ω, there exists a constant C_f such that

$$\sup_{x\in\Omega} \ln(tr((H_0)^{-1}(H_j(1))) + tr((H_j(1))^{-1} H_0)) \leq C_f. \qquad (44)$$

- Then we have uniform locally C^0-estimate for f_j.

- Since f_j is $\overline{\partial}_{0,j}$-holomorphic, then we have

$$\begin{aligned} \triangle_{0,j} f_j &= \sqrt{-1}\Lambda_\omega(\partial_{0,j}\overline{\partial_{0,j}} - \overline{\partial_{0,j}}\partial_{0,j}) f_j \\ &= -\sqrt{-1}\Lambda_\omega(\partial_{0,j}\overline{\partial_{0,j}} + \overline{\partial_{0,j}}\partial_{0,j}) f_j \\ &= -\sqrt{-1}\Lambda_\omega(F_{A_j} f_j - f_j F_{A_0}). \end{aligned} \tag{45}$$

- By the above uniform locally C^0 bound of f_j and the assumption that $A_j \to A_\infty$ in C^∞_{loc} topology outside Σ_{An}, the elliptic theory implies that there exists a subsequence of f_j (for simplicity, also denoted by f_j) such that $f_j \to f_\infty$ in C^∞_{loc} topology outside $\Sigma_{alg} \cup \Sigma_{An}$, and

$$\overline{\partial}_{A_0, A_\infty} f_\infty = 0, \quad f_\infty \circ \phi_0 = \phi_\infty \circ f_\infty. \tag{46}$$

- Furthermore, by (43) and the locally C^0-estimates (44), it is easy to conclude that f_∞ is non-zero.
- By Hartog's theorem, f_∞ extends to a non-zero Higgs sheaf homomorphism $f_\infty : (S, \phi_0) \to (E_\infty, \overline{\partial}_{A_\infty}, \phi_\infty)$ on M (where $(E_\infty, \overline{\partial}_{A_\infty}, \phi_\infty)$ is the extended reflexive Higgs sheaf).

Thank you!