A desingularization of the moduli space of Higgs bundles over a curve

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POSTECH

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History and Problem

- C = smooth projective curve over $\mathbb{C}$.
- $\mathcal{M}(r,d)$ = moduli space of semistable rank r Higgs bundles of degree d over C.
- $\mathbf{M}(r,\Lambda)$ = moduli space of semistable rank r Higgs bundles with fixed determinant Λ over C.
- $\mathbf{N}(r,\Lambda)$ = moduli space of semistable rank r vector bundles with fixed determinant Λ over C.
- History
  - (T.Hausel('98)) For a fixed line bundle Λ of degree 1 on C and a compactification $\overline{\mathbf{M}(2,\Lambda)}$ of $\mathbf{M}(2,\Lambda)$ as a symplectic quotient of $\mathbf{M}(2,\Lambda)\times\mathbb{C}$ by S^1,

$$\overline{\mathbf{M}(2,\Lambda)} \xleftarrow{\text{bir.}} \rightarrow \mathbb{P}(T^*\mathbf{N}(2,\Lambda)\oplus\mathcal{O}_C)$$

    by a sequence of blow-ups and blow-downs.

- History
  - (S.B.Bradlow, O.G-Prada, P.B.Gothen('04)) Birational descriptions of moduli spaces of α-stable holomorphic triples (when $g(C) \geq 2$) for a certain range of α.
  - (M.Mehta('05)) For the projective bundle $\mathcal{E}$ over $\mathcal{M}(r-1,d)$ with $\mathcal{E}|_{(E,\theta)} = \mathbb{P}\mathrm{Ext}^1((E,\theta),(\mathcal{O}_C,\mathrm{const.}))$,

$$\mathcal{E} \overset{\mathrm{bir.}}{\leftarrow - \rightarrow} \text{a fiber space over } \mathcal{M}_0(r,d)$$

    by a Morse theory on the space of τ-stable Higgs triples. Here,

$$\mathcal{M}_0(r,d) = \{(V,\phi) \in \mathcal{M}(r,d) |\, \det\phi = 0\}.$$

- Problem : Can we find a nonsingular moduli space birational to $\mathbf{M}(2,\mathcal{O}_C)$?
- Answer : Our work provides another way to construct a nonsingular variety birational to $\mathbf{M}(2,\mathcal{O}_C)$ with a moduli theoretic meaning.
- Y.-H. Kiem and J. Li successfully constructed a nonsingular variety birational to $\mathbf{N}(2,\mathcal{O}_C)$ in this way ('04).

Moduli space of semistable Higgs bundles

- C = smooth projective curve of genus $g \geq 3$ over $\mathbb{C}$.
- Higgs bundle is a pair (V, ϕ) of a rank 2 vector bundle $V \to C$ with $\det V = \mathcal{O}_C$ and $\phi \in H^0(\mathrm{End}_0 V \otimes K_C)$ where $\mathrm{End}_0 V$ is the bundle of trace-free endomorphisms of V.
- Higgs bundle (V, ϕ) is stable (resp. semistable) $\Leftrightarrow \forall$ subbundle $0 \neq W \lneq V$ satisfying $\phi(W) \subset W \otimes K_C$, $\deg W < 0$ (resp. $\deg W \leq 0$).
- Higgs bundle (V, ϕ) is polystable $\Leftrightarrow$ (V, ϕ) is stable or $(L, \psi) \oplus (L^{-1}, -\psi)$ for $L \in \mathrm{Pic}^0(C)$ and $\psi \in H^0(K_C)$.

- (N.Hitchin, N.Nitsure, C.Simpson)
  $\mathbf{M} := \mathbf{M}(2, \mathcal{O}_C) = \{\text{polystable Higgs bundles } (V, \phi) \text{ over } C/\cong\}$
  $= \mathbf{R}//SL(2)$
  $= \underline{\text{ moduli space of semistable Higgs bundles over } C}$
  is a quasi-projective irreducible normal variety of dimension $6g-6$.
  ($\mathbf{R}$ = quasi-projective irreducible normal variety of dimension $6g-3$ parametrizing $(V, \phi,\ \alpha : V|_x \xrightarrow{\cong} \mathbb{C}^2\ )$)
- (N.Hitchin)
  Stable Locus $\mathbf{M}^s$ in $\mathbf{M}$ is a smooth open dense subvariety of $\mathbf{M}$, equipped with a (holomorphic) symplectic form, i.e. $\mathbf{M}^s$ is hyperkähler.

Moduli space of stable parabolic Higgs bundles

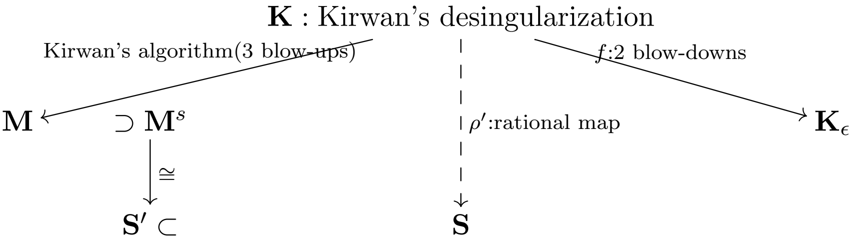
- Fix a point $x_0 \in C$.
- Parabolic Higgs bundle is a triple (E, ϕ, s) of a rank 4 vector bundle $E \to C$ with $\deg E = 0$, $\phi \in H^0(\mathrm{End} E \otimes K_C)$ and $s \in \mathbb{P}(E|_{x_0}^\vee)$ where ϕ is parabolic, that is, $\phi|_{x_0}(V(s)) \subset V(s)$.
- For real numbers $0 \le a_1 < a_2 < 1$, a parabolic Higgs bundle (E, ϕ, s) is stable (resp. semistable) with respect to the weights (a_1, a_2)
$\Leftrightarrow \forall$ parabolic subbundle $0 \neq (F, s_F) \lneq (E, s)$ satisfying $\phi((F, s_F)) \subset (F \otimes K_C, s_F)$,

$$\mathrm{par}\,\mu\,((F, s_F)) < \frac{a_1 + 3a_2}{4} \ (\text{resp. } \mathrm{par}\,\mu\,((F, s_F)) \le \frac{a_1 + 3a_2}{4}),$$

where
$$\mathrm{par}\,\mu\,((F, s_F)) = \begin{cases} \mu(F) + \frac{a_1 + (\mathrm{rk}(F) - 1)a_2}{\mathrm{rk}(F)} & \text{if } F|_{x_0} \not\subset V(s) \\ \mu(F) + a_2 & \text{if } F|_{x_0} \subset V(s) \end{cases}.$$

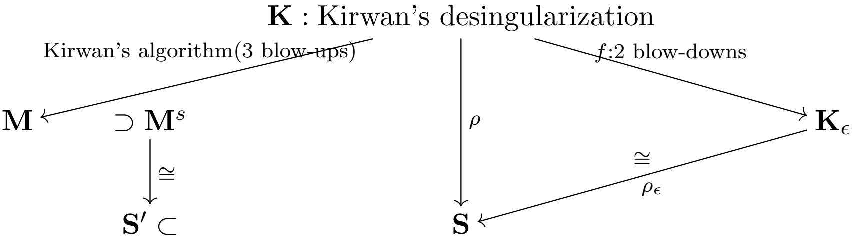
- Parabolic Higgs bundle (E, ϕ, s) is polystable with respect to the weights (a_1, a_2)
$\Leftrightarrow$ (E, ϕ, s) is stable or a direct sum of stable sub-triples with respect to the weights (a_1, a_2).
- (K. Yokogawa)
$\mathbf{M}^{\mathrm{par}}_{4,(a_1,a_2)} :=$ {polystable parabolic Higgs bundles (E, ϕ, s) with respect to the weights (a_1, a_2) over $C/\cong$}
$= \underline{\text{moduli space of semistable parabolic Higgs bundles}}$
$\underline{\text{with respect to the weights } (a_1, a_2) \text{ over } C}$
is a quasi-projective irreducible normal variety. For some small choice of (a_1, a_2), the stable locus $\mathbf{M}^{\mathrm{par},s}_{4,(a_1,a_2)}$ in $\mathbf{M}^{\mathrm{par}}_{4,(a_1,a_2)}$ coincides with $\mathbf{M}^{\mathrm{par}}_{4,(a_1,a_2)}$.

Main Result



K : Kirwan's desingularization
Kirwan's algorithm(3 blow-ups)
f:2 blow-downs
$\mathbf{M}$
$\supset \mathbf{M}^s$
$\cong$
ρ':rational map
$\mathbf{K}_\epsilon$
$\mathbf{S}' \subset$
$\mathbf{S}$

Main Result



Theorem(-)

1. $\mathbf{S}$ is nonsingular (main result). Further, there is a canonical morphism $\pi_{\mathbf{S}} : \mathbf{S} \to \mathbf{M}$ which is an isomorphism on $\mathbf{M}^s$.
2. ρ' is extended to a birational morphism $\rho : \mathbf{K} \to \mathbf{S}$.
3. ρ factors through $\mathbf{K}_\epsilon$. Furthermore, $\mathbf{K}_\epsilon \cong \mathbf{S}$.

- $\mathbf{S}$ is the Zariski closure of

$$\mathbf{S}' = \{[(E,\phi,s)] \in \mathbf{M}^{\mathrm{par},s}_{4,(a_1,a_2)} | \mathrm{End}((E,\phi)) \cong M(2), \det E = \mathcal{O}_C\}$$

where $M(2)$ is the $\mathbb{C}$-algebra of 2×2 matrices with entries in $\mathbb{C}$.
- For each $[(E,\phi,s)] \in \mathbf{S}$, $\mathrm{End}((E,\phi))$ is a specialization of $M(2)$, that is, a limit structure of $\mathbb{C}$-algebra structures of $M(2)$.
- $\mathbf{M}^s \xrightarrow{\cong} \mathbf{S}'$ is given by $(V,\phi) \mapsto ((V,\phi) \oplus (V,\phi), s_{\mathrm{can}})$. $\pi_{\mathbf{S}}|_{\mathbf{S}'}$ is the inverse of this isomorphism.
- $\pi_{\mathbf{S}} : \mathbf{S} \to \mathbf{M}$ is given by $(E,\phi,s) \mapsto (F,\psi)$, where (F,ψ) is determined by $Gr((E,\phi)) = (F,\psi) \oplus (F,\psi)$.
- $\mathbf{S}$ is a *desingularization* of $\mathbf{M}$.

Kirwan's algorithm

- $X(\subseteq \mathbb{P}^n)$: projective variety over $\mathbb{C}$.
- G = reductive complex algebraic group acting linearly on X, via a homomorphism
$$G \to GL(n+1).$$
- (Kirwan) Assume that $\exists x \in X^{ss} \setminus X^s$ such that $\dim \mathrm{Stab}(x)$ is (connected) maximal among stabilizers of points in $X^{ss} \setminus X^s$. Let $\pi : \tilde{X} \to X$ be the blow-up along $GZ^{ss}_{\mathrm{Stab}(x)}$, where $Z_{\mathrm{Stab}(x)}$ is the locus of fixed points by $\mathrm{Stab}(x)$ in X. Then

$$\begin{array}{ccc} \tilde{X}^{ss} & \longrightarrow & \tilde{X}//_{\tilde{L}}G \\ \pi \downarrow & \curvearrowright & \downarrow \overline{\pi} : \text{ blow-up of } X//G \text{ along } GZ^{ss}_{\mathrm{Stab}(x)}//G \\ X^{ss} & \longrightarrow & X//_L G \end{array} \qquad \text{with } \tilde{L} = \pi^* L^e \otimes \mathcal{O}(-E)(e >> 0)\,.$$

- (Kirwan) If $\exists y \in \tilde{X}^{ss} \setminus \tilde{X}^{s}$ such that $\dim \mathrm{Stab}(y)$ is (connected) maximal among stabilizers of points in $\tilde{X}^{ss} \setminus \tilde{X}^{s}$, then

$$\dim \mathrm{Stab}(y) < \dim \mathrm{Stab}(x).$$

- When $\dim \mathrm{Stab}(y) = 0$, Kirwan's algorithm terminates and $\tilde{X}^{ss} = \tilde{X}^{s}$. In this step, $\tilde{X}//_{\tilde{L}}G$ is called *Kirwan's partial desingularization* of $X//_{L}G$.

Kirwan's desingularization of $\mathbf{M}$ (along O'Grady's)

- Recall : $\mathbf{M} = \mathbf{R}//SL(2)$ ($\mathbf{R}$ is possibly singular).
- Every point in $\mathbf{R}$ is semistable with respect to the action of $SL(2)$. The closed orbits in $\mathbf{R}$ correspond to polystable Higgs bundles. The set $\mathbf{R}^s$ of stable points is exactly the locus of stable Higgs bundles.
- Kirwan's algorithm = a sequence of blow-ups of $\mathbf{R}$ along $\mathbf{R} \setminus \mathbf{R}^s$ until strictly semistable points disappear.
- $\mathbf{R} \setminus \mathbf{R}^s$ consists of
  (i) $(L, 0) \oplus (L, 0)$ for $L \cong L^{-1}$
  (ii) nontrivial extension of $(L, 0)$ by $(L, 0)$ for $L \cong L^{-1}$
  (iii) $(L, \psi) \oplus (L^{-1}, -\psi)$ for $(L, \psi) \not\cong (L^{-1}, -\psi)$
  (iv) nontrivial extension of $(L^{-1}, -\psi)$ by (L, ψ) for $(L, \psi) \not\cong (L^{-1}, -\psi)$

  where $L \in \mathrm{Pic}^0(X) =: J$ and $\psi \in H^0(K_X)$.
- Higgs bundles of type (ii) and (iv) are not polystable and their orbits do not appear in $\mathbf{M}$.
- $\mathrm{Stab}((\mathrm{i})) = SL(2)$ and $\mathrm{Stab}((\mathrm{iii})) = \mathbb{C}^*$.

- $\mathbf{M} = \mathbf{M}^s \sqcup (T^*J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}) \sqcup \mathbb{Z}_2^{2g}$
  - $\mathbf{M}^s$ =open subset of stable pairs
  - $T^*J/\mathbb{Z}_2 = \{(L,\psi) \oplus (L^{-1}, -\psi) | L \in \mathrm{Pic}^0(X), \psi \in H^0(K_X)\}/\mathbb{Z}_2$
  - $\mathbb{Z}_2^{2g} = \{(L,0) \oplus (L,0) | L \in \mathrm{Pic}^0(X), L \cong L^{-1}\}$
- (C.Simpson) Singularity along $\mathbb{Z}_2^{2g}$: $\mathbb{H}^g \otimes sl(2)///SL(2)$ where $SL(2)$ acts on $sl(2)$ by the adjoint action
- (C.Simpson) Singularity along $T^*J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}$: $\mathbb{H}^{g-1} \otimes \mathbb{C}^2///\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ with weights $1, -1$
- Kirwan's algorithm terminates in two steps as following :

$$\begin{array}{ccccc}
\mathbf{R}_2^{ss} = \mathbf{R}_2^s & \xrightarrow{\text{blow-up}/SL(2)(\widetilde{\text{iii}})} & \mathbf{R}_1^{ss} & \xrightarrow{\text{blow-up}/SL(2)(\text{i})} & \mathbf{R} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{R}_2^s/SL(2) & \xrightarrow{\text{blow-up}/(T^*J/\widetilde{\mathbb{Z}_2 - \mathbb{Z}}_2^{2g})} & \mathbf{R}_1//SL(2) & \xrightarrow{\text{blow-up}/\mathbb{Z}_2^{2g}} & \mathbf{M}_2 = \mathbf{R}//SL(2) \\
\uparrow & & & & \\
\vdots & & & & \\
\text{orbifold} & & & &
\end{array}$$

- $\mathbf{M}$ can be desingularized by three blow-ups along
  i) $\mathbb{Z}_2^{2g}$
  ii) proper transform of $T^*J/\mathbb{Z}_2$
  iii) nonsingular subvariety lying in the proper transform of the exceptional divisor of the first blow-up
  $\Rightarrow$ Kirwan's desingularization $\mathbf{K} \to \mathbf{M}$ with exceptional divisors D_1, D_2 and D_3.

Elemantary modification of Higgs bundles

- $(\mathcal{E}, \varphi) =$ a family of Higgs bundles parametrized by a scheme T.
  $\iota : D \hookrightarrow T$: Cartier divisor.
  $(\mathcal{F}, \varphi_{\mathcal{F}}) \subset (\mathcal{E}, \varphi)|_{C \times D}$: a family of sub-Higgs bundles on D.
  $(\mathcal{G}, \varphi_{\mathcal{G}})$: a family of quotients.
  Elementary modification $(\mathcal{E}', \varphi')$ of $(\mathcal{E}, \varphi)$ is given by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \iota_*\mathcal{G} & \longrightarrow & 0\ , \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \iota_*\varphi_{\mathcal{G}}} & & \\ 0 & \longrightarrow & \mathcal{E}' \otimes pr_1^* K_C & \longrightarrow & \mathcal{E} \otimes pr_1^* K_C & \longrightarrow & \iota_*\mathcal{G} \otimes pr_1^* K_C & \longrightarrow & 0 \end{array}$$

  where horizontal maps are short exact sequences.

Construction of ρ (over D_2)

- For $l = [(L, \psi) \oplus (L^{-1}, -\psi)] \in T^*J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}$,

$$\mathcal{N}_l := H^1(\mathcal{O}_C) \oplus H^0(K_C) \oplus Q$$

  where Q is a smooth quadratic hypersurface in $\mathbb{C}^{4g-4}$, which is the fiber of the normal cone of the locus of type (iii) in $\mathbf{R}$.

- By Luna's slice theorem, $\exists$ family $(\mathcal{F}, \varphi_{\mathcal{F}})$ of semistable Higgs bundles parametrized by a neighborhood U_l of 0 in $\mathcal{N}_l$ (From the universal object over $C \times \mathbf{R}$) such that

$$(\mathcal{F}, \varphi_{\mathcal{F}})|_{C \times (U_l \cap (H^1(\mathcal{O}_C) \oplus H^0(K_C)))} = (\mathcal{L}, \psi_{\mathcal{L}}) \oplus (\mathcal{L}^{-1}, -\psi_{\mathcal{L}}),$$

  where $(\mathcal{L}, \psi_{\mathcal{L}})$ is a family of Higgs bundles of rank 1 and degree 0.

- $\pi_l : \tilde{U}_l \to U_l$: blow-up of U_l along $U_l \cap (H^1(\mathcal{O}_C) \oplus H^0(K_C))$ with the exceptional divisor D_l.
- $(\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}}) := \pi_l^*(\mathcal{F}, \varphi_{\mathcal{F}})$ and $(\tilde{\mathcal{L}}, \psi_{\tilde{\mathcal{L}}}) := \pi_l^*(\mathcal{L}, \psi_{\mathcal{L}})$
  $\Rightarrow \exists$ surjective $(\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}})|_{D_l} \to (\tilde{\mathcal{L}}, \psi_{\tilde{\mathcal{L}}})$ and $(\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}})|_{D_l} \to (\tilde{\mathcal{L}}^{-1}, -\psi_{\tilde{\mathcal{L}}})$.
- (Elementary modifications)
  $(\tilde{\mathcal{F}}', \varphi_{\tilde{\mathcal{F}}'}) := \ker((\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}}) \to (\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}})|_{D_l} \to (\tilde{\mathcal{L}}, \psi_{\tilde{\mathcal{L}}}))$
  and
  $(\tilde{\mathcal{F}}'', \varphi_{\tilde{\mathcal{F}}''}) := \ker((\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}}) \to (\tilde{\mathcal{F}}, \varphi_{\tilde{\mathcal{F}}})|_{D_l} \to (\tilde{\mathcal{L}}^{-1}, -\psi_{\tilde{\mathcal{L}}}))$
- $(\mathcal{E}, \varphi_{\mathcal{E}}) := (\tilde{\mathcal{F}}', \varphi_{\tilde{\mathcal{F}}'}) \oplus (\tilde{\mathcal{F}}'', \varphi_{\tilde{\mathcal{F}}''})$ is a family parametrized by $\tilde{U}_l$ such that $\mathrm{End}((\mathcal{E}, \varphi_{\mathcal{E}})|_{C \times \{t\}})$ is a specialization of $M(2)$ on an open dense subset of $\tilde{U}_l$.
  $\Rightarrow (\mathcal{E}, \varphi_{\mathcal{E}})$ gives us a continuous extension of ρ' to a neighborhood of the points in $\mathbf{K}$ over l.
  $\Rightarrow$ This extension is indeed a morphism.
  (By GAGA and Riemann's extension theorem)

- Note that the transitions and local Higgs fields of $(\mathcal{E}, \varphi_{\mathcal{E}})$ are of the form

$$\left(\begin{pmatrix}\lambda_{ij} & b_{ij} & 0 & 0\\ 0 & \lambda_{ij}^{-1} & 0 & 0\\ 0 & 0 & \lambda_{ij} & 0\\ 0 & 0 & c_{ij} & \lambda_{ij}^{-1}\end{pmatrix}, \begin{pmatrix}p_i & q_i & 0 & 0\\ 0 & -p_i & 0 & 0\\ 0 & 0 & p_i & 0\\ 0 & 0 & r_i & -p_i\end{pmatrix}\right)$$

Specialization of $M(2)$

- C_n=unitary associative algebra generated by a 3-dimensional complex vector space V with a quadratic form q of rank $n(\leq 3)$ subject to $uv + vu = q(u+v) - q(u) - q(v)$ for all $u, v \in V$.
- C_n is $\mathbb{Z}_2$-graded. Indeed, $V \to V(v \mapsto -v)$ extends to an involution $\alpha : C_n \to C_n$. Thus, we have the eigenspace decomposition $C_n = C_n^+ \oplus C_n^-$.
- $C_3^+ \cong M(2)$.
- (Along V.Balaji's arguments) $\mathbf{S}_3 \subset \mathbf{S}_2 \subset \mathbf{S}_1 \subset \mathbf{S}$
  - $\mathbf{S} - \mathbf{S}_1 = \{[(E,\phi,s)] \in \mathbf{S} | \mathrm{End}((E,\phi)) \cong C_3^+ \cong M(2)\} = \pi_{\mathbf{S}}^{-1}(\mathbf{M}^s)$
  - $\mathbf{S}_1 - \mathbf{S}_2 = \{[(E,\phi,s)] \in \mathbf{S} | \mathrm{End}((E,\phi)) \cong C_2^+\} = \pi_{\mathbf{S}}^{-1}(T^*J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g})$ is a smooth divisor of $\mathbf{S} - \mathbf{S}_2 = \mathbf{S} - \pi_{\mathbf{S}}^{-1}(\mathbb{Z}_2^{2g})$.
  - $\mathbf{S}_2 - \mathbf{S}_3 = \{[(E,\phi,s)] \in \mathbf{S} | \mathrm{End}((E,\phi)) \cong C_1^+\}$
  - $\mathbf{S}_3 = \{[(E,\phi,s)] \in \mathbf{S} | \mathrm{End}((E,\phi)) \cong C_0^+\}$

- C_2^+ has generators x, w, u, v with the following defining relations

$$u^2 = u, v^2 = v, uv = 0, u + v = 1,$$
$$w^2 = x^2 = wx = xw = 0, uw = w, wu = 0,$$
$$ux = 0, xu = x, vw = 0, wv = w,$$
$$vx = x, xv = 0.$$

- C_1^+ has generators $1, y_1, y_2, y_3$ with the following defining relations

$$y_1^2 = y_2^2 = y_3^2 = y_1y_3 = y_2y_3 = 0, \quad y_1y_2 = -y_2y_1 = y_3.$$

- C_0^+ has generators y_1, y_2, y_3 with the following defining relations

$$y_1^2 = y_2^2 = y_3^2 = y_1y_2 = y_1y_3 = y_2y_3 = 0.$$

- Fix $e_0 \in \mathbb{C}^4$. Let $\mathcal{A}(2)$ be the set of elements in $\mathrm{Hom}(\mathbb{C}^4 \otimes \mathbb{C}^4, \mathbb{C}^4)$ which gives us an algebra structure on $\mathbb{C}^4$ with the identity element e_0. There is a subset of $\mathcal{A}(2)$ which consists of algebra structures on $\mathbb{C}^4$, isomorphic to $M(2)$. Let $\mathcal{A}_2$ be the closure of this subset. An element in $\mathcal{A}_2$ is called a *specialization of* $M(2)$.

Nonsingularity of $\mathbf{S}$, The proof of Theorem-1

- The morphism of functors $\Phi : Mor(-, \mathbf{S}) \to Mor(-, \mathcal{A}_2)$ is formally smooth, that is, $\forall$ small surjection $p : A' \to A$ of local artinian commutative $\mathbb{C}$-algebras with residue field $\mathbb{C}$ and

$$\begin{array}{ccc} \exists \gamma \in Mor(\mathrm{Spec}\, A', \mathbf{S}) & \xrightarrow{\Phi_{A'}} & \forall \beta \in Mor(\mathrm{Spec}\, A', \mathcal{A}_2) \\ {\scriptstyle Mor(-,\mathbf{S})(p)} \big\downarrow & \curvearrowright & \big\downarrow {\scriptstyle Mor(-,\mathcal{A}_2)(p)} \\ \forall \alpha \in Mor(\mathrm{Spec}\, A, \mathbf{S}) & \xrightarrow{\Phi_{A}} & Mor(\mathrm{Spec}\, A, \mathcal{A}_2) \end{array}$$

- To determine γ, we should classify transitions and local Higgs fields of underlying Higgs bundles of all objects in $\mathbf{S}_1$. For this, we should use the defining relations of C_2^+, C_1^+ and C_0^+.

Proposition(-)

Let $(E, \phi, s) \in \mathbf{S}_1$. Then the underlying Higgs bundle (E, ϕ) is given by one of the following type of pairs of transitions and local Higgs fields :

$$\text{(I)} \quad \left(\begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ 0 & \lambda_0^{-1} & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_0^{-1} \end{pmatrix}, \begin{pmatrix} p_0 & p_1 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & p_2 & -p_0 \end{pmatrix} \right)$$

$$\text{(II)} \quad \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_2 & 1 & 0 & 0 \\ -\lambda_1 & 0 & 1 & 0 \\ \lambda_0 & \lambda_1 & \lambda_2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 \\ -p_1 & 0 & 0 & 0 \\ p_0 & p_1 & p_2 & 0 \end{pmatrix} \right)$$

$$\text{(III)} \quad \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda_0 & \lambda_1 & \lambda_2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_0 & p_1 & p_2 & 0 \end{pmatrix} \right).$$

- (The key of the proof of (I))
  The left regular representation of C_2^+ is given as follows :
  For $a = \lambda_0 u + \lambda_1 x + \lambda_2 w + \lambda_3 v$,

$$a \cdot u = \lambda_0 \cdot u + \lambda_1 \cdot x + 0 \cdot w + 0 \cdot v$$

$$a \cdot x = 0 \cdot u + \lambda_3 \cdot x + 0 \cdot w + 0 \cdot v$$

$$a \cdot w = 0 \cdot u + 0 \cdot x + \lambda_0 \cdot w + 0 \cdot v$$

$$a \cdot v = 0 \cdot u + 0 \cdot x + \lambda_2 \cdot w + \lambda_3 \cdot v.$$

  Then the left regular representation of A is given by

$$\lambda_0 u + \lambda_1 x + \lambda_2 w + \lambda_3 v \mapsto \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 \end{pmatrix}$$

- (The key of the proof of (II))
  The left regular representation of C_1^+ is given by

$$(\lambda_0 + \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3) \mapsto \begin{pmatrix} \lambda_3 & 0 & 0 & 0 \\ \lambda_2 & \lambda_3 & 0 & 0 \\ -\lambda_1 & 0 & \lambda_3 & 0 \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

- (The key of the proof of (III))
  The left regular representation of C_0^+ is given by

$$(\lambda_0 y_1 + \lambda_1 y_2 + \lambda_2 y_3 + \lambda_3) \mapsto \begin{pmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

- $\mathcal{A}_2$ is nonsingular, that is, $Mor(-, \mathcal{A}_2)(p)$ is surjective.
  $\Rightarrow Mor(-, \mathbf{S})(p)$ is surjective, that is, $\mathbf{S}$ is nonsingular.

The proof of Theorem-2 and Theorem-3

- Theorem-2 : Construct a nice family of parabolic Higgs bundles of rank 4 by successive applications of elementary modifications of Higgs bundles, beginning with a family of rank 2 Higgs bundles from the universal object over $\mathbf{R}$.
- Theorem-3 :
  - (O'Grady) $f : \mathbf{K} \to \mathbf{K}_\epsilon$ is the composition of contractions of $K_{\mathbf{K}}$-negative extremal rays in D_3 and D_1, and $\mathbf{K}_\epsilon$ is nonsingular.
  - (O'Grady)
$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{\rho} & \mathbf{S} \\ & {\scriptstyle f}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle \rho_\epsilon} & \\ & \mathbf{K}_\epsilon & \end{array}.$$
  - $\rho_\epsilon : \mathbf{K}_\epsilon \to \mathbf{S}$ is an isomorphism in codimension one since $\mathbf{S}_1 - \mathbf{S}_2$ is a smooth divisor of $\mathbf{S} - \mathbf{S}_2$..
  - $\rho_\epsilon : \mathbf{K}_\epsilon \to \mathbf{S}$ is an isomorphism from the Zariski's main theorem.

Further question

- Consider the first contraction $\mathbf{K} \to \mathbf{K}_\sigma$ of $K_{\mathbf{K}}$-negative extremal ray in D_3.
- Question : Find the closest nonsingular moduli space to $\mathbf{K}_\sigma$.
- Known : In the case of $\mathbf{N}(2, \mathcal{O}_C)$, $\mathbf{K}_\sigma(\mathbf{N}(2, \mathcal{O}_C))$ is isomorphic to the *moduli space of Hecke cycles* associated to stable vector bundles. (Choe, Choy and Kiem, '05).
- Revised question : Construct a moduli space of Hecke cycles associated to stable Higgs bundles and compare it with $\mathbf{K}_\sigma$.

Thank you