Approximations of weighted Ehrhart polynomials and generating functions by intermediate sums on polyhedra

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This is a report on joint work with V.Baldoni, J. De Loera, M. Koeppe and M. Vergne. For a given semi-rational convex polytope $\mathfrak{p} \subset \mathbb{R}^d$ and a weight $h(x)$ (polynomial function on $\mathbb{R}^d$), we study the sum of values $h(x)$ over the set of lattice points in the dilated polytope $t\mathfrak{p}$, for a real parameter t. This sum is a quasi-polynomial $E(t)$ of degree $d + \deg h$, called the weighted Ehrhart quasi-polynomial of $\mathfrak{p}$ with respect to the weight h.

We compute two approximations of this quasi-polynomial, that is, two quasi-polynomials which coincide with $E(t)$ in degree $\geq d + \deg h - k$ for a fixed k. They are defined in terms of intermediate sums, interpolating between integrals and discrete sums, which were introduced by A. Barvinok [Computing the Ehrhart quasi-polynomial of a rational simplex, 2006]. The first approximation is the same as Barvinok's, while the second one is new and simpler.

We introduce the corresponding intermediate generating function of a polyhedron P, with respect to a rational subspace $L \subset \mathbb{R}^d$,

$$S^L(P)(\xi) = \sum_y \int_{P\cap(y+L)} \mathrm{e}^{\langle \xi, x\rangle}\,\mathrm{d}x.$$

where we integrate over the slices of P (intersection of P with subspaces parallel to L through all lattice points), and sum the integrals.

This defines a meromorphic function of ξ which enjoys an additive property with respect to the polyhedron P. We prove an approximation theorem for these generating functions, in the sense of power series, from which we derive the approximations of weighted Ehrhart quasi-polynomials.

Our results are based on a thorough analysis of the intermediate generating functions of a shifted affine polyhedral cone, when the vertex of the cone is considered as a parameter.