Algebraic Geometry and Convex Geometry

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Intersection index on an irreducible variety X, $\dim X = n$

Let $K(X)$ be the semigroup of spaces L of rational functions on X such that: a) $\dim L < \infty$, and b) $L \neq 0$.

For $L_1, L_2 \in K(X)$, the product is the space $L_1 L_2 \in K(X)$ generated by elements fg, where $f \in L_1$, $g \in L_2$.

For $L_1, \ldots, L_n \in K(X)$, the intersection index $[L_1, \ldots, L_n]$ is

$\# x \in X : (f_1(x) = \cdots = f_n(x) = 0)$, where $f_1 \in L_1, \ldots, f_n \in L_n$ is a generic n-tuple of functions. We neglect roots $x \in X$ such that $\exists i : (f \in L_i \Rightarrow f(x) = 0)$, and such that $\exists f \in L_j$ for $1 \leq j \leq n$ having a pole at x.

The intersection index is multi-linear with respect to the product in $K(X)$.

Let $X = (\mathbb{C}^*)^n$, let $A \subset (\mathbb{Z})^n$ be a finite set; let $L_A \in K(X)$ be the space generated x^m, where $m \in A$; $\Delta(A)$ be the convex hull of A and $V(\Delta(A))$ be its volume.

Kushnirenko's theorem (1975)

$[L_A, \ldots, L_A] = n!V(\Delta(A))$.

Why?

Why convex hull?

Why volume?

Mixed volume

$(\exists\,!)$ $V(\Delta_1,\ldots,\Delta_n)$, on n-tuples of convex bodies in $\mathbb{R}^n$, such that:

1. $V(\Delta,\ldots,\Delta)$ is the volume of Δ;
2. V is symmetric;
3. V is multi-linear; for example, $V(\Delta_1'+\Delta_1'',\Delta_2,\ldots)=V(\Delta_1',\Delta_2,\ldots)+V(\Delta_1'',\Delta_2,\ldots)$;
4. $\Delta_1'\subseteq\Delta_1,\ldots,\Delta_n'\subseteq\Delta_n\Rightarrow V(\Delta_1',\ldots,\Delta_n')\leq V(\Delta_1,\ldots,\Delta_n)$;
5. $0\leq V(\Delta_1,\ldots,\Delta_n)$.

Bernstein's theorem (1975)

$[L_{A_1},\ldots,L_{A_n}]=n!V(\Delta(A_1),\ldots,\Delta(A_n))$.

Grothendieck semigroup Gr(S)

For a commutative semigroup S let $a \sim b \Leftrightarrow (\exists c \in S)|(a + c = b + c)$.

Then $\mathrm{Gr}(S)$ is $S/\sim$. Let $\rho : S \to \mathrm{Gr}(S)$ be the natural map.

The Grothendieck group of S is the group of formal differences of $\mathrm{Gr}(S)$.

Theorem 1. *Let $\mathcal{K}$ be the semigroup of finite subsets $\mathbb{Z}^n \subset \mathbb{R}^n$ with respect to addition. Then $\mathrm{Gr}(\mathcal{K})$ consists of convex polyhedra and $\rho(A)$ is the convex hull $\Delta(A)$ of A.*

The index $[L_1, \ldots, L_n]$ can be extended to the Grothendieck group $\mathrm{Gr}(K(X))$ of $K(X)$ and considered as a birationally invariant generalization of the intersection index of divisors, which is applicable to non-complete varieties.

The group $G(K(X))$ of $K(X)$

One can describe the relation $\sim$ in $K(X)$ as follows: $f \in \mathbb{C}(X)$ is called integral over L if it satisfies an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0$$

with $m > 0$ and $a_i \in L^i$. The collection of all integral functions over L is a finite-dimensional subspace $\overline{L}$ called the completion of L. In $K(X)$:

1. $L_1 \sim L_2 \Leftrightarrow \overline{L}_1 = \overline{L}_2$;

2. $L \sim \overline{L}$;

3. $L \sim M \Rightarrow M \subset \overline{L}$.

Semigroup of integral points, its Newton–Okounkov body

Let $S \subset \mathbb{Z}^n$ be a semigroup,
$G(S) \subset \mathbb{Z}^n$ the group generated by S;
$L(S) \subset \mathbb{R}^n$ the subspace spanned by S;
$C(S) = \overline{(\text{convex hullof } S \cup \{0\})}$.
The regularization $\tilde{S}$ of S is the semigroup $C(S) \cap G(S)$.

Theorem 2. *Let $C' \subset C(S)$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin. Then there exists a constant $N > 0$ (depending on C') such that any point in the group $G(S)$ which lies in C' and whose distance from the origin is bigger than N belongs to S.*

Let $M_0 \subset L(S)$ be a space; $\dim M_0 = \dim L(S) - 1 = q$; $C(S) \cap M_0 = 0$;

Let M_k be the affine space parallel to M_0 and intersection G which has distance k from the origin (the distance is normalized in such a way that as values it takes all the non-negative integers k).

The Hilbert function H_S of S is define by $H_S(k) = \# M_k \cap S$.

The Newton–Okounkov body $\Delta(S)$ of S is define by $\Delta(S) = C(S) \cap M_1$.

Theorem 3. *The function $H_S(k)$ grows like $a_q k^q$ where q is the dimension of the convex body $\Delta(S)$, and the q-th growth coefficient a_q is equal to the (normalized in the appropriate way) q-dimensional volume of $\Delta(S)$.*

Algebra of almost finite type, its Newton–Okounkov body

Let F be a field of transcendence degree n over $\mathbf{k}$. Let $F[t]$ be the algebra of polynomials over F. We deal with graded subalgebras in $F[t]$:

1. $A_L = \bigoplus_{k \geq 0} L^k t^k$, where $L \subset F$ is a subspace, $\dim_{\mathbf{k}} L < \infty$; $L^0 = \mathbf{k}$ and L^k is the span of all the products $f_1 \cdots f_k$ with $f_1, \ldots, f_k \in L$.
2. An algebra of almost integral type is a subalgebra in some algebra A_L.

We construct a $\mathbb{Z}^{n+1}$-valued valuation v_t on $F[t]$ by extending a $\mathbb{Z}^n$-valuation v on F which takes all the values in $\mathbb{Z}^n$.

Let $\Delta(A)$ be the Newton–Okounkov body of the semigroup $S(A) = v_t(A \setminus \{0\})$ projected to $\mathbb{R}^n$ (via the projection on the first factor $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$).

Theorem 4. *1. The Hilbert function $H_A(k)$ of A grows like $a_q k^q$, where q is an integer between 0 and n.*

2. $q = \dim_{\mathbb{R}} \Delta(A)$, and a_q is the (normalized in the appropriate way) q-dimensional volume of $\Delta(A)$.

One defines a componentwise product of graded subalgebras. Consider the class of graded algebras of almost integral type such that, for $k \gg 0$, all their k-th homogeneous components are non-zero. Let A_1, A_2 be algebras of such kind and put $A_3 = A_1 A_2$. It is easy to verify the inclusion $\Delta(A_1) + \Delta(A_2) \subset \Delta(A_3)$.

Brunn–Minkowsky inequality in convex geometry

$$V^{1/n}(\Delta_1) + V^{1/n}(\Delta_2) \leq V^{1/n}(\Delta_1 + \Delta_2).$$

Theorem 5. $a_n^{1/n}(A_1) + a_n^{1/n}(A_2) \leq a_n^{1/n}(A_3)$.

Newton–Okounkov bodies and Intersection theory

With a space $L \in K(X)$, we associate the algebra A_L and its integral closure $\overline{A_L}$ in the field $\mathbb{C}(X)[t]$ and two corresponding bodies $\Delta(A_L) \subseteq \Delta(\overline{A_L})$.

For a big space L we have $\Delta(A_L) = \Delta(\overline{A_L})$.

Theorem 6. *For $L \in K(X)$ we have:*

1. $[L, \dots, L] = n!\mathrm{Vol}(\Delta(\overline{A_L}))$.

2. $\Delta(\overline{A_{L_1 L_2}}) \supseteq \Delta(\overline{A_{L_1}}) + \Delta(\overline{A_{L_2}})$

Proof. Follows from the theorem 4. □

The Kušnirenko theorem is a special case of the theorem 6. The Newton polyhedron of the product of two Laurent polynomials is equal to the sum of the corresponding Newton polyhedra. This property gives the Bernstein theorem as a corollary of the theorem 6.

Theorem 7. *Let* $L_1, L_2 \in K(X)$ *and* $L_3 = L_1L_2$. *We have:* $[L_1, \dots, L_1]^{1/n} + [L_2, \dots, L_2]^{1/n} \leq [L_3, \dots, L_3]^{1/n}$.

Hodge type inequality. For $n = 2$ we have $[L_1, L_1][L_2, L_2] \leq [L_1, L_2]^2$.

Alexandrov–Fenchel type inequality in algebra and geometry

Alexandrov–Fenchel inequality in convex geometry

$$V^2(\Delta_1, \Delta_2, \ldots, \Delta_n) \geq V(\Delta_1, \Delta_1, \ldots, \Delta_n)V(\Delta_2, \Delta_2, \ldots, \Delta_n).$$

Theorem 8. *Let* X, $\dim X = n$, *be an irreducible variety, let* $L_1, \ldots, L_n \in K(X)$ *and let* $L_3, \ldots, L_n$ *be big subspaces. Then* $[L_1, L_2, L_3, \ldots, L_n]^2 \geq$

$$[L_1, L_1, L_3, \ldots, L_n][L_2, L_2, L_3, \ldots, L_n].$$

The Alexandrov–Fenchel inequality in convex geometry follow easily from the theorem 8 via the Bernstein–Kušnirenko theorem. This trick has been known. Our contribution is an elementary proof of the key analogue of the Hodge index inequality which makes all the chain of arguments involved elementary and more natural.

Other results

1. **local intersection theory**. Let R_a be the ring of germs of regular functions at a point $a \in X, \dim X = n$.

   Let $\mathbf{K}_a$ be the set of ideals $L \subset R_a$ of finite co-dimension, $\dim_{\mathbb{C}} R_a/L < \infty$. For $L_1, \ldots, L_n \in \mathbf{K}_a$ the local intersection index $[L_1, \ldots, L_n]_a$ is defined: it is equal to the multiplicity at the origin of a system $f_1 = \cdots = f_n = 0$, where f_i is a generic function from L_i.

   Theorem 9. ***(local algebraic Alexandrov–Fenchel type inequality)*** *Let* $L_1, \ldots, L_n \in \mathbf{K}_a$. *Then* $[L_1, L_2, \ldots, L_n]_a^2 \leq [L_1, L_1, \ldots, L_n]_a[L_2, L_2, \ldots, L_n]_a$.

2. **Local geometric version** Let $C \subset \mathbb{R}^n$ be a strongly convex cone. A compact set $A \subset C$ is called co-convex body if $C \setminus A$ is convex. Put $A \oplus B = C \setminus [(C \setminus A) + C \setminus B]$.

   The set of co-convex bodies with the operation $\oplus$ is a comutative semigroup. It's Grothendieck group is a real vector space $BL(C)$.

Let V_C be the homogeneous degree n polynomial on $BL(C)$ such that $V_C(A)$ is equal to the volume of a co-convex body A. The mixed volume $V_C(A_{i_1}\dots,A_{i_n})$ of co-convex sets $A_{i_1}\dots,A_{i_n})$ is the value of the polarization of V_C on the n-tuple $A_{i_1},\dots,A_{i_n}$.

Theorem 10. ***(Local Alexandrov–Fenchel inequality)*** $V_C(A_1,A_2,\dots,A_n)^2 \leq V_C(A_1,A_1,\dots,A_n)V_C(A_2,A_2,\dots,A_n)$.

3. For $L \in \mathcal{K}$ the Newton–Okounkov body $\Delta(\overline{A_L})$ strongly depends on a choice of $\mathbb{Z}^n$-valued valuation v on $\mathbb{C}(X)$. If X is equipped with a reductive group action and if one is interested only in the invariant subspaces $L \in \mathcal{K}$, then one can use the freedom to make all results more precise and explicit

4. Another result of the theory: one can prove analogues of Fujita approximation theorem for semigroups of integral points and

graded algebras, which implies a generalization of this theorem for arbitrary linear series.

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