On probability measures with algebraic Cauchy transform

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Basic definitions

The logarithimic potential and the Cauchy transform of a (compactly supported complex-valued) measure μ are given by

$$\mathfrak{u}_\mu(z) := \int_{\mathbb{C}} \ln|z-\xi| d\mu(\xi)$$

and

$$\mathcal{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z-\xi} = \frac{\partial \mathfrak{u}_\mu(z)}{\partial z}$$

.

The knowledge of (a germ of) $\mathcal{C}_\mu(z)$ near ∞ is equivalent to the knowledge of its (holomorphic) moment sequence $m_k^\mu, k = 0, 1, ...$ where

$$m_k^\mu = \int_{\mathbb{C}} z^k d\mu(z).$$

More precisely, the Taylor expansion of $\mathcal{C}_\mu(z)$ at ∞ has the form:

$$\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots.$$

History

The study of local and global properties of the Cauchy transform and the Cauchy-Stieltjes integral was initiated by A. Cauchy and T. Stieltjes in the middle of the 19th century.

Dozens of well-known mathematicians contribute to this topic over more than a century. Numerous papers and several books are partially or completely devoted to this area.It is closely connected with the (logarithmic) potential theory and, especially, its inverse problem and inverse moment problem.

Motherbody

During the last 2 decades the notion of a motherbody of a solid domain or, more generally, of a positive Borel measure was discussed in the geophysics and mathematics literature.

This notion was apparently pioneered by in the 60's by a Bulgarian geophysicist D. Zidarov and mathematically developed by B. Gustafsson. Although several interesting results about this notion were obtained there still seems to be no consensus even about the definition of a motherbody and no general existence and uniqueness results are known.

Below we use one of possible definitions of a motherbody and address a quite natural (exterior) inverse motherbody problem in potential theory which has not yet attracted that much of attention. (In what follows we will always consider Borel measures.)

Main problem

Main problem. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 \in \mathbb{R}$ of an algebraic (or, more generally, analytic) function near ∞ is it possible to find a compactly supported in $\mathbb{C}$ signed measure whose Cauchy transform coincides with (a branch of) the analytic continuation of $f(z)$ a.e. in $\mathbb{C}$? Additionally, for which $f(z)$ it is possible to find a positive measure with the above properties?

By a signed measure we mean the difference between two positive measures. If such a signed measure exists we will call it *a (real) motherbody measure* of the germ $f(z)$. If the corresponding measure is positive then we call it *a positive motherbody measure* of $f(z)$. An obvious necessary condition for the existence of a positive motherbody measure is that $a_0 > 0$. A germ (branch) of an analytic function $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 > 0$ will be called *positive*. If $a_0 = 1$ then $f(z)$ is called a *probability branch*. Necessary and sufficient conditions for the existence of a probability branch of a given algebraic function are given below.

Basic example

The standard measure $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in [-1, 1]$ has the Cauchy transform

$$\mathcal{C}(z) = \frac{1}{\sqrt{z^2 - 1}}, \quad z \notin [-1, 1].$$

In other words,

$$(z^2 - 1)\mathcal{C}^2 = 1$$

a.e. in $\mathbb{C}$.

Main definition

A formal definition of a motherbody measure we use below is as follows.

Definition

Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 > 0$ of an analytic function near ∞ we say that a signed measure μ_f is its *motherbody* if:

(i) its support $supp(\mu_f)$ is the union of compact semi-analytic curves in $\mathbb{C}$;

(ii) The Cauchy transform of μ_f coincides in each connected component of the complement $\mathbb{C} \setminus supp(\mu_f)$ with some branch of the analytic continuation of $f(z)$.

Sometimes we will additionally assume that each connected component of the support of a motherbody measure has a singular point.

Some remarks

Notice that by a general result of Borcea-Bøgvad-Björk from 2010 if one requires that if the Cauchy transform of a given positive measure coincides with a given (irreducible) analytic function a.e. in $\mathbb{C}$ then its support should be a finite union of semi-analytic curves.

In general, the inverse problem in potential theory deals with the question how to restore a solid body or a (positive) measure from the information about its potential near infinity. The main efforts in this inverse problem since the foundational paper of P. S.Novikov were concentrated around the question about the uniqueness of a (solid) body with a given exterior potential. P. S. Novikov (whose main mathematical contributions are in the areas of mathematical logic and group theory) proved uniqueness for convex and star-shaped bodies.

More remarks

The question about the uniqueness of contractible domains in $\mathbb{C}$ with a given sequence of holomorphic moments was later posed anew by H. S. Shapiro and answered negatively by M. Sakai. A similar non-uniqueness example for non-convex plane polygons was reported by geophysicists M. A. Brodsky and V. N. Strakhov but, in fact, it was discovered by A. Gabrielov (who attended this program earlier).

Important observation

It turns out that in $\mathbb{C}$ the existence of a positive measure with a given Cauchy transform $f(z)$ near ∞ imposes no conditions on a germ except for the obvious $a_0 > 0$. Namely, our first result is as follows.

Lemma

Given an arbitrary germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$, $a_0 > 0$ of an analytic function near ∞ there exist (a plenty of) positive compactly supported in $\mathbb{C}$ measures whose Cauchy transform near ∞ coincides with $f(z)$.

Proof. Given a branch $f(z) = a_0/z + \sum_{i\geq 2} a_i/z^i$ of an analytic function near ∞ we first take a germ of its 'logarithmic potential', i.e. a germ $h(z)$ of harmonic function such that $h(z) = a_0 \log |z| + \dots$ where $\dots$ stands for a germ of harmonic function near ∞ and satisfying the relation $\partial h/\partial z = f(z)$ in a punctured neighborhood of ∞.

For any sufficiently large positive v the level curve $\gamma_v : h(z) = v$ near infinity is a closed and simple curve making one turn around ∞. To get a required positive measure whose Cauchy transform coincides with $f(z)$ near ∞ take γ_v for any v large enough and consider the complement $\Omega_v = \mathbb{C}P^1 \setminus Cl(\gamma_v)$ where $Cl(\gamma_v)$ is the interior of γ_v containing ∞. Consider the equilibrium measure of mass a_0 supported on Ω_v. By Frostman's theorem, this measure is supported on γ_v (since γ_v has no polar subsets), its potential is constant on γ_v and this potential is harmonic in the complement to the support. Thus it should coincide with $h(z)$ in $Cl(\gamma_v)$ since the total mass is correctly chosen. Then the Cauchy transforms coincide in $Cl(\gamma_v)$ as well. □

On the other hand, the requirement that the Cauchy transform coincides with (the analytic continuation) of a given germ $f(z)$ a.e. in $\mathbb{C}$ often imposes additional restrictions on the germ $f(z)$ which in certain special cases are presented below.

Case of rational functions

Lemma

A (germ at ∞ of a) rational function $\mathcal{C}(z) = \frac{z^n + \dots}{z^{n+1} + \dots}$ with coprime numerator and denominator admits a real motherbody measure if and only if it has all simple poles with real residues. If all residues are positive the corresponding motherbody measure is positive.

Proof. Recall the classical relation between a measure μ and its Cauchy transform $\mathcal{C}_\mu$. Namely one has

$$\frac{\partial \mathcal{C}_\mu(z)}{\partial z} = \mu$$

where the derivative is taken in the sense of distributions. Since by assumptions $\mathcal{C}_\mu(z)$ coincides almost everywhere with a given rational function which is univalent. Therefore, μ is the measure supported at its poles and coinciding up to a factor 2π with the residues.

cont.

Since by assumption μ has to be a real measure this implies that all poles of the rational function should be simple and with real residues. □

Remark. The above statement implies that the set of rational function of degree *n* admitting a real motherbody measure has the dimension equals to the half of the dimension of the space of all rational functions of the above form of degree *n*.

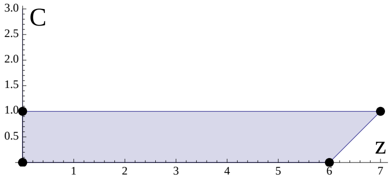


Figure : Newton polygon of a rational function with a probability branch.

Necessary conditions

We first give a necessary and sufficient condition for an algebraic curve given by the equation $P(\mathcal{C}, z) = \sum_{(i,j)\in S} \alpha_{i,j}\mathcal{C}^i z^j = 0$ to have a probability branch at ∞. Here S is an arbitrary finite subset of pairs of non-negative integers, i.e an arbitrary set of monomials in 2 variables. (In what follows $\mathcal{C}$ will stand for the variable corresponding to the Cauchy transform.) Given a polynomial $P(\mathcal{C}, z)$ let $S(P)$ denote the set of all its monomials, i.e. monomials appearing in P with non-vanishing coefficients.

Lemma

The curve given by the equation $P(\mathcal{C}, z) = \sum_{(i,j)\in S(P)} \alpha_{i,j}\mathcal{C}^i z^j = 0$ *has a probability branch at* ∞ *if and only if* $\sum_i \alpha_{i,M(P)-i} = 0$ *where* $M(P) = \min_{(i,j)\in S(P)} i - j$. *In particular, there should be at least two distinct monomials in* $S(P)$ *whose difference of indices equals* $M(P)$.

Newton polygons

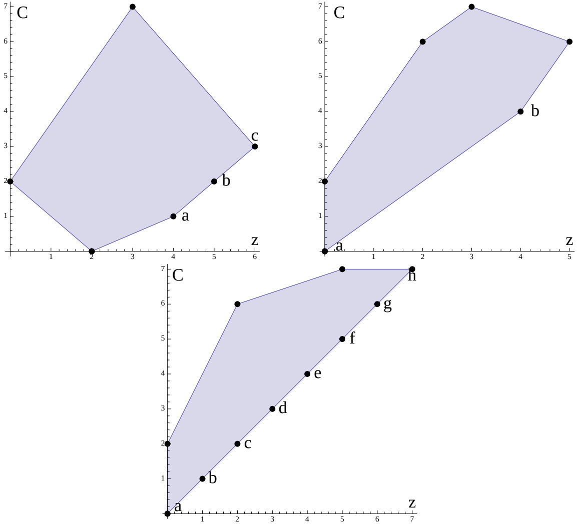


Figure : Three examples of Newton polygons.

Let S be an arbitrary set of monomials in 2 variables, Pol_S be the linear span of these monomials with complex coefficients, and $\mathcal{N}_S$ be the *Newton polygon* of S, i.e. the convex hull of S.

Lemma

A generic polynomial from Pol_S is irreducible if and only if

(i) S contains a vertex on both coordinate axes, i.e. pure powers of both variables;

(ii) if the only monomial in S on coordinates axes is the origin then S should contain at least two non-proportional monomials, i.e. not lying on the same ray through the origin.

Corollary

An irreducible polynomial $P(\mathcal{C}, z)$ having a probability branch has $M(P) = \min_{(i,j)\in S(P)} i - j$ non-negative. If $M(P) = 0$ then the set $S(P)$ of monomials of $P(\mathcal{C}, z)$ must contain the origin, i.e., a non-vanishing constant term.

Main Conjecture

If we denote by $\mathcal{N}_P = \mathcal{N}_{S(P)}$ the Newton polygon of $P(\mathcal{C}, z)$ then the geometric interpretation of the latter corollary is that $\mathcal{N}_P$ should (non-strictly) intersect the bisector of the first quadrant on the plane $(z, \mathcal{C})$. Now we present our main conjecture and supporting results.

Main Conjecture. An arbitrary irreducible polynom $P(\mathcal{C}, z)$ with a positive branch and $M(P) = 0$ admits a positive motherbody measure.

Appropriate Newton polygons are shown on the central and the right-most pictures on the above Figure.

Main Result I

The next result essentially proven by Bøgvad-Borcea-S strongly supports the above conjecture. Further supporting statements were earlier found by T. Bergkvist.

Theorem

An arbitrary irreducible polynomial $P(\mathcal{C}, z)$ with a probability branch and $M(P) = 0$ which additionally satisfies two requirements:

(i) *$S(P)$ contains a diagonal monomial $\mathcal{C}^k z^k$ which is lexicographically bigger than any other monomial in $S(P)$;*

(ii) *the probability branch is 'simple';*

(iii) *the probability branch is the only positive branch of $P(\mathcal{C}, z)$;*

admits a probability motherbody measure.

By 'lexicographically bigger' we mean that the pair (k, k) is coordinate-wise not smaller than any other pair of indices in $S(P)$ as shown on the right-most picture on the above Figure.

Illuminating example

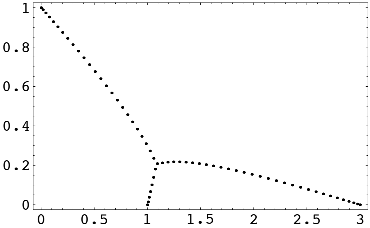


Figure : Roots of an eigenpolynomial of the operator $T = (z-1)(z-3)(z-I)\frac{d^3}{dz^3}$

The Cauchy transform $\mathcal{C}_\mu$ of the limiting root-counting measure satisfies a.e. the equation $(z-1)(z-3)(z-I)\mathcal{C}_\mu^3 = 1$.

More complicated example

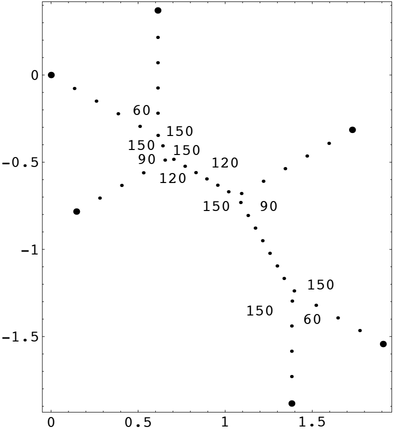


Figure : Roots of an eigenpolynomial of an operator of order 6

Sketch of the proof

Given an algebraic function given by the polynomial equation $P(\mathcal{C}, z) = \sum_{(i,j)\in S} \alpha_{i,j}\mathcal{C}^i z^j = 0$ consider the differential operator

$$T_P = \sum_{(i,j)\in S} \alpha_{i,j} z^j \frac{d^i}{dz^i} = \sum_{i=0}^{k} Q_i(z) \frac{d^i}{dz^i}.$$

By the above assumptions $\deg Q_i(z) \leq i,\ i = 0, 1, ..., k$ and $\deg Q_k(z) = k$. Consider the following (homogenized polynomial) spectral problem. For a given non-negative integer n find λ such that there exists a polynomial $p(z)$ of degree n solving:

$$Q_k(z)p^{(k)} + \lambda Q_{k-1}(z)p^{(k-1)} + \lambda^2 Q_{k-2}(z)p^{(k-2)} + ... + \lambda^k Q_0 p = 0.$$

Sketch of the proof, cont.

Results from Publ. RIMS vol. 45 (2) (2009) 525–568.

Lemma

Under our assumptions there exist a sequence $\{\lambda_n\}$ $(n \geq N_P)$ *of eigenvalues such that the corresponding eigenpolynomials* $p_n(z)$ *are of degree* n *and* $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$.

Lemma

There exists an open disk centered at ∞ *which contains no roots of* $p_n(z)$ *for all* n *simultaneously.*

cont. 2

Theorem

In the above notation there exists its subsequence $\{p_{i_n}(z)\}$ *of the above sequence* $\{p_n(z)\}$ *such that the (sub)sequence* $\left\{\frac{p'_{i_n}(z)}{\lambda_{i_n} p_{i_n}(z)}\right\}$ *converges almost everywhere in* $\mathbb{C}$ *to a limiting function* $\mathcal{C}$ *then* $\mathcal{C}$ *which satisfies a.e. in* $\mathbb{C}$ *the symbol equation*

$$Q_k(z)\mathcal{C}^k + Q_{k-1}(z)\mathcal{C}^{k-1} + \ldots + Q_0\mathcal{C} = 0.$$

The question about the convergence of the whole sequence still remains widely open.

Illustration

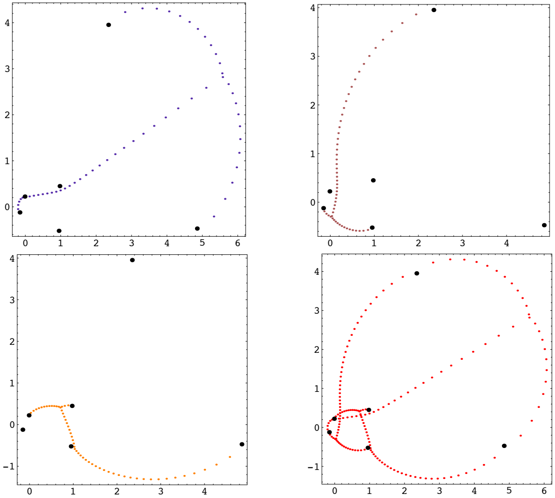


Figure : Three root-counting measures for the spectral pencil $T_\lambda = \left(z^3 - (5+2I)z^2 + (4+2I)z\right)\frac{d^3}{dz^3} + \lambda(z^2 + Iz + 2)\frac{d^2}{dz^2} + \lambda^2 \frac{1}{5}(z - 2 + I)\frac{d}{dz} + \lambda^3.$

What about uniqueness?

Uniqueness of positive motherbody measures does not hold, in general, even for the simplest algebraic equations of the form

$$z(1-z)\mathcal{C}^2 + ((1-B)z + C)\mathcal{C} + B = 0$$

with arbitrary complex parameters B and C.

Lemma

The support of the motherbody measure must belongs to the set of horizontal trajectories of the quadratic differential

$$\Psi(z) = -\frac{(1+B)^2z^2 + 2(C - CB - 2B)z + C^2}{z^2(1-z)^2}dz^2.$$

The positive motherbody measure is unique if and only the above differentials has no circular domains which happens for generic values of B and C.

Suspicion. If one requires that each connected component of the support of a positive motherbody measure has a singularity then the positive measure (if exists) is unique. Similarly, if one requires that each connected component of the support of a real motherbody measure has a singularity then there exists only finitely many such measures.

What happens for polynomials with $M(P) > 0$

Let S be an arbitrary set of monomials in 2 variables, Pol_S be the linear span of these monomials with complex coefficients, and $\mathcal{N}_S$ be the *Newton polygon* of S, i.e. the convex hull of S. Assume that there are at least two monomials with the minimal difference $(i - j)$ and let $\widetilde{Pol}_S$ be the linear subspace of Pol_S satisfying the necessary condition of the existence of the probability measure.

Main Conjecture. In the linear space $\widetilde{Pol}_S$ with $M(P) > 0$ the subset of polynomials admitting a positive motherbody measure is a real semi-analytic variety of codimension $M(P)$.

If S has degree at least 2 in $\mathcal{C}$ then the subset of polynomials admitting a real motherbody measure is a countable union of real semi-analytic variety of codimension $M(P)$ which is dense in $\widetilde{Pol}_S$.

Square roots and quadratic differentials

Here we consider in details what happens for the algebraic functions satisfying $Q(z)\mathcal{C}^2 = P(z)$ where $Q(z) = z^k + \dots$ and $P(z) = z^{k+2} + \dots$, i.e. $\mathcal{C} = \sqrt{R(z)}$ where $R(z) = \frac{z^k + \dots}{z^{k+2} + \dots}$.

Definition. A (meromorphic) quadratic differential Ψ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the tensor square $(T^*_{\mathbb{C}}Y)^{\otimes 2}$ of the holomorphic cotangent bundle $T^*_{\mathbb{C}}Y$. The zeros and the poles of Ψ constitute the set of *singular points* of Ψ denoted by $Sing_\Psi$. (Non-singular points of Ψ are usually called *regular*.)

If Ψ is locally represented in two intersecting charts by $h(z)dz^2$ and by $\tilde{h}(\tilde{z})d\tilde{z}^2$ resp. with a transition function $\tilde{z}(z)$, then $h(z) = \tilde{h}(\tilde{z})\,(d\tilde{z}/dz)^2$. Any quadratic differential induces a canonical metric on its Riemann surface, whose length element in local coordinates is given by

$$|dw| = |h(z)|^{\frac{1}{2}}|dz|.$$

The above canonical metric $|dw| = |h(z)|^{\frac{1}{2}}|dz|$ on Y is closely related to two distinguished line fields given by the condition that $h(z)dz^2$ is either positive or negative. The first field is given by $h(z)dz^2 > 0$ and its integral curves are called *horizontal trajectories* of Ψ, while the second field is given by $h(z)dz^2 < 0$ and its integral curves are called *vertical trajectories* of Ψ. In what follows we will mostly use horizontal trajectories of rational quadratic differentials and reserve the term *trajectories* for the horizontal ones.

A trajectory of a meromorphic quadratic differential Ψ given on a compact Y without boundary is called *singular* if there exists a singular point of Ψ belonging to its closure. A non-singular trajectory $\gamma_{z_0}(t)$ of a meromorphic Ψ is called *closed* if $\exists\ T > 0$ such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$. The least such T is called the *period* of γ_{z_0}.

A quadratic differential Ψ on a compact Riemann surface Y without boundary is called *Strebel* if the set of its closed trajectories covers Y up to a set of Lebesgue measure zero.

For a given quadratic differential $\Psi \in \mathcal{M}$ on a compact surface Y denote by $K_\Psi \subset Y$ the union of all its singular trajectories and singular points.

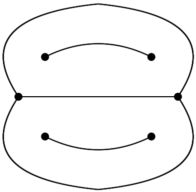
It is known that for a meromorphic Strebel differential Ψ given on a compact Riemann surface Y without boundary the set K_Ψ has several nice properties. In particular, it is a finite embedded multigraph on Y whose edges are singular trajectories connecting pairs of singular points of Ψ. (Here by a *multigraph* on a surface we mean a graph with possibly multiple edges and loops.)

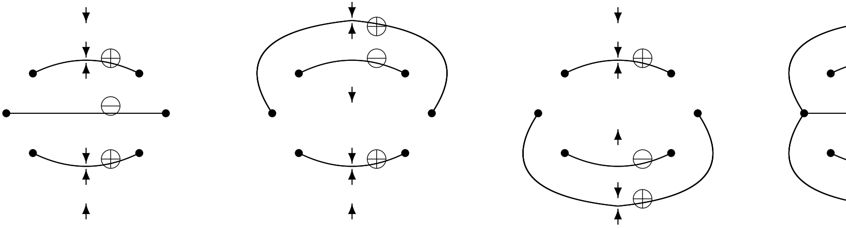
Theorem

Given two coprime monic polynomials $P(z)$ and $Q(z)$ of degrees n and $n+2$ resp. where n is a non-negative integer one has that the algebraic function given by the equation

$$Q(z)\mathcal{C}^2 = P(z) \tag{1}$$

admits a real motherbody measure $\mu_{\mathcal{C}}$ if and only if the quadratic differential $\Psi = -P(z)dz^2/Q(z)$ is Strebel. Such motherbody measures are, in general, non-unique but the support of each such measure consists of singular trajectories, i.e. is a subgraph of K_Ψ.





1st measure   2nd measure   3rd measure   4th r

Concerning possible positive measures we can formulate an exact criterion of the existence of a positive measure for a rational quadratic differential $\Psi = -P(z)dz^2/Q(z)$ in terms of rather simple topological properties of K_Ψ.

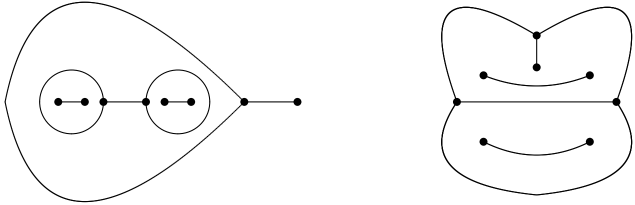
The vertices of K_Ψ are the finite singular points of Ψ (i.e. excluding ∞) and its edges are singular trajectories connecting these finite singular points. Each (open) connected component of $\mathbb{C} \setminus K_\Psi$ is homeomorphic to an (open) annulus. K_Ψ might have isolated vertices which are the finite double poles of Ψ. Vertices of K_Ψ having valency 1 (i.e. hanging vertices) are exactly the simple poles of Ψ. Vertices different from the isolated and hanging vertices are the zeros of Ψ. The number of edges adjacent to a given vertex minus 2 equals the order of the zero of Ψ at this point. Finally, the sum of the multiplicities of all poles (including the one at ∞) minus the sum of the multiplicities of all zeros equals 4.

By a *simple cycle* in a planar multigraph K_Ψ we mean any closed non-selfintersecting curve formed by the edges of K_Ψ.

Proposition.

A Strebel differential $\Psi = -P(z)dz^2/Q(z)$ admits a positive motherbody measure if and only if no edge of K_Ψ is attached to a simple cycle from inside. In other words, for any simple cycle in K_Ψ and any edge not in the cycle but adjacent to some vertex in the cycle this edge does not belong to its interior. The support of the positive measure coincides with the forest obtained from K_Ψ after the removal of all its simple cycles.

Remark. Notice that under the above assumptions all simple cycles of K_Ψ are pairwise non-intersecting and, therefore, their removal is well-defined in an unambiguous way.



Examples of K_Ψ admitting and not admitting a positive measure

The right picture admits no positive measure since it contains an edge cutting a simple cycle (the outer boundary) in two smaller cycles. The left picture has no such edges and, therefore, admits a positive measure whose support consists of the four horizontal edges of K_Ψ.

THANK YOU FOR YOUR ATTENTION!!!