Discrete convolution operators, the Fourier transformation, and its tropical counterpart: the Fenchel transformation

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Abstract

We study solvability of convolution equations for functions with discrete support in $\mathbb{R}^n$, a special case being functions with support in the integer points. The more general case is of interest for several grids in Euclidean space, like the body-centered and face-centered tesselations of three-space.

Abstract

We study solvability of convolution equations for functions with discrete support in $\mathbb{R}^n$, a special case being functions with support in the integer points. The more general case is of interest for several grids in Euclidean space, like the body-centered and face-centered tesselations of three-space.

The theorem of existence of fundamental solutions by Boor, Höllig & Riemenschneider is generalized to general discrete supports using only elementary methods. We also study the asymptotic growth of sequences and arrays using the Fourier and Fenchel transformations.

Introduction

Many sequences and arrays are defined recursively, like

$$
\begin{aligned}
f(x) &= a_1 f(x-1) + a_2 f(x-2) + \cdots + a_m f(x-m), \\
&\quad x \in \mathbb{N},\ x \geqslant x_0; \\
f(x,y) &= a_{1,0} f(x-1,y) + a_{0,1} f(x,y-1) + a_{1,1} f(x-1,y-1) + \cdots \\
&\quad + a_{m,m} f(x-m,y-m), \\
&\quad (x,y) \in \mathbb{N}^2,\ x \geqslant x_0,\ y \geqslant y_0,
\end{aligned}
$$

typically with some initial conditions. These sequences and arrays can conveniently be described as solutions to convolution equations on $\mathbb{Z}$ and $\mathbb{Z}^2$, respectively.

The purpose here is to study convolution equations of the general form $\nu * w = \rho$, where w is the unknown function, and where ν and ρ are given functions defined on $\mathbb{R}^n$ and of finite support—sometimes we shall relax the latter condition. We thus go from functions on $\mathbb{Z}^n$ (the most studied discretization) to more general functions.

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This allows, for instance, discretizations corresponding to other tessellations of $\mathbb{R}^n$, like the body-centered cubic (bcc) grid and the face-centered cubic (fcc) grid in $\mathbb{R}^3$ studied by Strand (2008) and others. These are periodic, but coming to quasicrystals, we must allow for non-periodic functions.

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In fact, there is a scale of regularity, starting with $\mathbb{Z}^n$ as the most regular set and ending with arbitrary discrete sets. Somewhere between these are the quasicrystals. How can we measure this regularity?

We shall study the solutions with the help of the Fourier and Fenchel transformations. Infimal convolution and the Fenchel transformation can be viewed as tropicalizations of usual convolution and the Fourier (or Laplace) transformation, respectively—tropicalization is in itself a most interesting transformation.

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In particular we shall prove that convolution equations have fundamental solutions, a result proved by de Boor, Höllig & Riemenschneider (1989). Our method of proof is elementary, while theirs relies on a modification to the discrete case of Hörmander's proof (1958) of the division theorem for distributions, which in turn builds on the Tarski–Seidenberg theorem. See also Łojasiewicz (1958, 1959).

Our result is more general, since we allow finite supports consisting of arbitrary points in $\mathbb{R}^n$, not necessarily integer points, and also some infinite discrete supports.

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We get a fundamental solution with support in a strict convex cone and in general with exponential growth there, while the solution of de Boor et al. is of polynomial growth but with support spread out. There is often a trade-off between growth and information on the support; in this way our solutions are more like the solutions to hyperbolic equations and more suited to initial-value problems.

Notation

We shall use $\mathbb{R}_+$ to denote the set of all positive real numbers and $\mathbb{R}_! = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$ to denote the set of extended real numbers, adding two infinities.

Addition $\mathbb{R}^2 \ni (x, y) \mapsto x + y \in \mathbb{R}$ can be extended in two different ways to operations $(\mathbb{R}_!)^2 \to \mathbb{R}_!$: the ***upper sum*** $x \dot{+} y$ is defined as $+\infty$ if one of the terms is equal to $+\infty$, and the ***lower sum*** $x \underset{\cdot}{+} y$ is defined as $-\infty$ if one of the terms is equal to $-\infty$. We use $x \wedge y$ for the minimum of x and y; $x \vee y$ for the maximum. Under these operations $\mathbb{Z}$ and $\mathbb{R}$ are lattices, and $\mathbb{Z}_!$ and $\mathbb{R}_!$ complete lattices.

The ***indicator function*** $\mathbf{ind}_A = -\log \chi_A$, where χ_A is the characteristic function, will be used.

We shall use the l^p-norm $\|x\|_p = (\sum_j |x_j|^p)^{1/p}$, $1 \leqslant p < +\infty$, and the l^∞-norm $\|x\|_\infty = \sup_j |x_j|$ for $x \in \mathbb{R}^n$. We shall use these norms also for functions, e.g., $\|f\|_1 = \sum_x |f(x)|$. When any norm can serve, we write only $\|x\|$. The inner product is written $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$, $(\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

In a metric space X with metric d we shall denote by $B_{<}(c,r)$ and $B_{\leqslant}(c,r)$ the ***open ball*** and the ***closed ball*** with center at $c \in X$ and radius $r \in \mathbb{R}$, respectively, thus

$$B_{<}(c,r) = \{x \in X;\ d(x,c) < r\} \text{ and } B_{\leqslant}(c,r) = \{x \in X;\ d(x,c) \leqslant r\}.$$

The closure and interior of a subset A of a topological space will be denoted by $\overline{A}$ and A°, respectively. Thus in $\mathbb{R}^n$, $\overline{B_{<}(c,r)} = B_{\leqslant}(c,r)$ if r is positive, and $B_{\leqslant}(c,r)^{\circ} = B_{<}(c,r)$ for all real r.

Classes of discrete sets

Discrete sets and uniformly discrete sets

Following Bourbaki (1961:16) we shall say that a subset A of a metric space X with metric d is ***discrete*** if for each point $a \in A$, a is the only point in $B_<(a, r)$ for some positive r. It is called ***uniformly discrete*** if r can be chosen independently of a.

For any subset A of X we define a function

$$\mathrm{dist}_A(x) = \inf_a(d(a, x);\ a \in A,\ a \neq x), \qquad x \in X.$$

Thus A is discrete iff dist_A is positive everywhere, and A is uniformly discrete iff its restriction $(\mathrm{dist}_A)|_A$ to A is bounded from below by a positive constant.

Given a measurable set $C \subset \mathbb{R}^n$ with $0 < \mathbf{vol}(C) < +\infty$, we define the ***mean density in*** C of a subset A of $\mathbb{R}^n$ as the quotient

$$\frac{\mathbf{card}(A \cap C)}{\mathbf{vol}(C)}.$$

The ***upper*** and ***lower density*** of a subset A of $\mathbb{R}^n$ is defined as the upper and lower limit, respectively, of the mean density in $C = B_{\leqslant}(0, r)$ as r tends to $+\infty$. The upper density of a uniformly discrete set in $\mathbb{R}^n$ is clearly finite.

Uniformly spanning sets

Another property is

$$\exists r \in \mathbb{R}\ \forall x \in X\ \exists a \in A\ \ d(a,x) \leqslant r.$$

We shall say that A is ***uniformly spanning*** in this case. Equivalently, dist_A is bounded. In $\mathbb{R}^n$, this means that $A + B_{\leqslant}(0,r) = \mathbb{R}^n$ for some r.

Clearly the lower density of a uniformly spanning set in $\mathbb{R}^n$ is positive.

Delone sets

A set A is said to be a ***Delone set*** (named for Борис Николаевич Делоне, 1890–1980) if is uniformly discrete and uniformly spanning. This means that there exist positive numbers r_0 and r_1 such that

$$\mathbf{card}(B_{\leqslant}(c, r_0) \cap A) \leqslant 1 \leqslant \mathbf{card}(B_{\leqslant}(c, r_1) \cap A), \qquad c \in X.$$

Quasicrystals are, or rather define by their locations, Delone sets.

Temperate discreteness

Maybe it is of interest to define in $\mathbb{R}^n$ temperate versions of the properties just defined: we shall say that $A \subset \mathbb{R}^n$ is ***temperately discrete*** if dist_A is positive and $((\operatorname{dist}_A)|_A)^{-1}$ is bounded from above by a polynomial. It follows that the mean density in $B_{\leqslant}(0, r)$ is bounded from above by a polynomial in r.

We shall also consider the larger class of sets such that $1/\operatorname{dist}_A(a)$, $a \in A$, grows slower than any exponential function $e^{\varepsilon\|a\|}$, for instance like $e^{\|a\|^\alpha}$ for some α, $0 < \alpha < 1$.

Asymptotic density

Also for uniformly spanning sets we have a weaker version in $\mathbb{R}^n$:

$$\bigcup_{a \in A} B_{\leqslant}(a, \varphi(a)) = \mathbb{R}^n,$$

for some function φ such that $\varphi(x)/\|x\|$ tends to zero as $\|x\| \to +\infty$. We shall say that A is ***asymptotically dense*** if this is the case.

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Equivalently, dist_A is bounded from above by such a function φ.

Combining the two, we obtain a more general kind of Delone sets, the ***temperate Delone sets***, defined by the requirement that

$$\mathbf{card}(B_{\leqslant}(c, r_0(c)) \cap A) \leqslant 1 \leqslant \mathbf{card}(B_{\leqslant}(c, r_1(c)) \cap A), \qquad c \in \mathbb{R}^n,$$

where r_0^{-1} is bounded by a polynomial and $r_1(c)/\|c\|$ tends to zero as $\|c\| \to +\infty$.

Convolution

Let G be an abelian group—most of the time we shall take $G = \mathbb{Z}^n$ or $G = \mathbb{R}^n$. We define the ***convolution product*** $h = f * g$ of two functions $f, g \colon G \to \mathbb{C}$ by

$$h(x) = \sum_{y+z=x} f(y)g(z), \qquad x \in G,$$

provided the sum is finite for all x. We can define three kinds of algebras satisfying this provision.

The ***Kronecker delta*** δ_a, defined by $\delta_a(a) = 1$ and $\delta_a(x) = 0$ for $x \neq a$, satisfies $\delta_a * \delta_b = \delta_{a+b}$. We shall write just δ for δ_0, which is a neutral element for convolution: $f * \delta = f$ for all functions f.

Case α'. First we have the algebra $\mathbb{C}^{[G]}$ of all functions $G \to \mathbb{C}$ with finite support. (The support of a function is here just the set where it is nonzero.)

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We always have

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g, \qquad f, g \in \mathbb{C}^{[G]},$$

in general with a strict inclusion. However, if f, g are nonnegative, or more generally if the set $\{f(y)g(z);\ y, z \in G\}$ of all products of values of f and g is contained in a strict convex cone in the complex plane, then we have equality:

$$\operatorname{supp}(f * g) = \operatorname{supp} f + \operatorname{supp} g, \qquad f, g \in \mathbb{C}^{[G]}.$$

When $G = \mathbb{R}^n$ or $\mathbb{Z}^n$, we have

$$\mathbf{cvxh}(\mathrm{supp}(f * g)) = \mathbf{cvxh}(\mathrm{supp}\, f) + \mathbf{cvxh}(\mathrm{supp}\, g), \qquad f, g \in \mathbb{C}^{[\mathbb{R}^n]},$$

where $\mathbf{cvxh}(A)$ denotes the convex hull of a set A. This is easily proved using induction over the dimension. It is a precise quantitative form of the fact that the algebra does not have zero divisors in this case. For some groups, like the cyclic groups $G = \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, the algebra $\mathbb{C}^{[G]}$ does have zero divisors.

The equation for the convex hulls of the supports corresponds to the Titchmarsh support theorem for compactly supported continuous functions or distributions on $\mathbb{R}^n$, which, however, does not have an easy proof.

The algebra $\mathbb{C}^{[\mathbb{Z}^n]}$ is isomorphic to the algebra of Laurent polynomials:

$$P_f(z) = \sum_{x \in \mathbb{Z}^n} f(x) z^x = \sum_{x \in \mathbb{Z}^n} f(x) z_1^{x_1} \cdots z_n^{x_n}, \qquad z \in \mathbb{C}^n,$$

and also to the algebra of trigonometric polynomials, putting $z_j = e^{i\zeta_j}$:

$$\hat{f}(\zeta) = \sum_{x \in \mathbb{Z}^n} f(x) e^{i\zeta \cdot x}, \qquad \zeta \in \mathbb{C}^n.$$

Case β'. For $G = \mathbb{R}^n$, given a nonzero vector $\theta \in \mathbb{R}^n$, we consider the algebra $\mathscr{A}_\theta$ of all functions with discrete support contained in $\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant 0\}$ and such that, for any $r > 0$, there are only finitely many points in the support of f in the half space $\{x \in \mathbb{R}^n;\ \theta \cdot x \leqslant r\}$.

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To make this tractable, we introduce the set Φ of all functions $\varphi\colon\ [0,+\infty[\to [0,+\infty[$ which are superadditive: $\varphi(s) + \varphi(t) \leqslant \varphi(s+t)$, $s, t \geqslant 0$. We can for example take $\varphi(t) = t^\alpha$, $\alpha \geqslant 1$, or $\varphi(t) = t\psi(t)$, where ψ is a convex and increasing function, e.g., $\psi(t) = e^{t^\alpha}$, $\alpha \geqslant 1$. We then take f with support contained in the set

$$V_\theta^\varphi = \{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant 0,\ \|x\| \leqslant \varphi(\theta \cdot x)\}, \qquad \theta \in \mathbb{R}^n,\ \varphi \in \Phi.$$

We note that $V_\theta^\varphi + V_\theta^\varphi = V_\theta^\varphi$. When $\varphi(t) = t$ we write

$$V_\theta^{\mathrm{id}} = V_\theta = \{x \in \mathbb{R}^n;\ \|x\| \leqslant \theta \cdot x\}, \qquad \theta \in \mathbb{R}^n,$$

a strict convex cone.

The product $f(y)g(x-y)$ is nonzero only when both y and $x-y$ belong to V_θ^φ, which implies that $y \in V_\theta^\varphi \cap (x - V_\theta^\varphi)$, a bounded set for any given x, thus for only finitely many y.

The convolution product of two functions with support in V_θ^φ has its support in V_θ^φ: if $x \notin V_\theta^\varphi$, then, for every y, either $y \notin V_\theta^\varphi$ or $x-y \notin V_\theta^\varphi$. Thus we get convolution algebras $\mathscr{A}_\theta^\varphi$ for every φ and θ; the algebra $\mathscr{A}_\theta$ is the union of all the $\mathscr{A}_\theta^\varphi$ when φ varies.

Case γ'. For $G = \mathbb{R}^n$, the functions with discrete support in a translate of a set V_θ^φ also form an algebra. When $n = 1$ it is even a field. We can of course also take the union of V_θ^φ over all φ.

However, sometimes we need to define a convolution product in other situations.

Case δ'. We can define a convolution product $f_1 * \cdots * f_k$ when all factors except one have finite support.

Case ε'. For $G = \mathbb{R}^n$, we can define a convolution product $f_1 * \cdots * f_k$ when all factors except one have their support, assumed to be discrete, contained in translates of a set V_θ^φ and the remaining one has its support, also discrete, contained in a half space

$$\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant s\}$$

for some real number s and with the same vector θ. Also here we can take the union over all $\varphi \in \Phi$.

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for some real number s and with the same vector θ. Also here we can take the union over all $\varphi \in \Phi$.

In these four cases, the associative law holds. However, associativity is a subtle property and can easily be lost:

Example

Take $f(x) = 1$ for all $x \in \mathbb{Z}$; $g = \delta_{-1} - \delta_0$ (a difference operator); and $h(x) = 1$ for all $x \in \mathbb{N}$, $h(x) = 0$ for $x \leqslant -1$.

Then $f * g = 0$ (Case δ') and $(f * g) * h = 0$, while $g * h = \delta_{-1}$ (Case δ') and $f * (g * h) = 1 \neq 0$.

Note that neither $f * h$ nor $f * g * h$ here can be defined in accordance with any of the Cases α'–ε'. $\square$

We shall study in particular convolution equations of the form $\nu * w = \rho$, where ν and ρ have finite support (Case δ'), and also when ν has its support in some V_θ^φ.

Case ζ'. We can also define a convolution product in some other cases where the sum is infinite and has a good convergence.

Solving convolution equations

Theorem

Let $\nu, \rho : \mathbb{R}^n \to \mathbb{C}$ be functions with finite support, $\nu \neq 0$. Then there is a function $w : \mathbb{R}^n \to \mathbb{C}$ with discrete support which solves the equation $\nu * w = \rho$.

Since $\rho = \sum_a \rho(a)\delta_a$, it is enough to solve $\nu * w_a = \delta_a$ for $a \in \operatorname{supp} \rho$ and then form the finite linear combination $w = \sum_a \rho(a) w_a$.

We first find a normal form for these equations.

Lemma

Given ν, ρ of finite support and any point $b \in \operatorname{supp} \nu$, define for any function w,

$$u = (\nu(b)w) * \delta_{b-a} \text{ and } \mu = -\sum_{x \neq b} \nu(x)\nu(b)^{-1}\delta_{x-b}.$$

Then the equation

$$\nu * w = \delta_a$$

has a solution w if and only if u solves the equation

$$(\delta - \mu) * u = \delta.$$

Moreover, if b is a vertex of $\mathbf{cvxh}(\operatorname{supp} \nu)$*, then 0 does not belong to the convex hull of $\operatorname{supp} \mu$.*

Proof.

The equation $\nu * w = \delta_a$ can be written

$$\delta_b * (\nu(b)w) + \sum_{x \neq b} (\nu(x)\nu(b)^{-1}\delta_x) * (\nu(b)w) = \delta_a,$$

equivalently

$$\delta * (\nu(b)w * \delta_{b-a}) + \sum_{x \neq b} (\nu(x)\nu(b)^{-1}\delta_{x-b}) * (\nu(b)w * \delta_{b-a}) = \delta.$$

We can now introduce u and μ as indicated. $\square$

We note that the support of μ is contained in a strict convex cone:

Lemma

Given any finite set A in $\mathbb{R}^n$ such that the origin does not belong to the convex hull of A, there exists a vector $\theta \in \mathbb{R}^n$ and a strict convex cone K such that

$$A \subset K \cap \{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant 1\}.$$

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$$A \subset K \cap \{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant 1\}.$$

Proof.

In view of the Hahn–Banach theorem there exists a vector θ such that every $x \in A$ satisfies $\theta \cdot x \geqslant 1$. We can then take the cone $V_{s\theta}$, where $s = \sup_{a \in A} \|a\| (\theta \cdot a)^{-1}$, provided $A \neq \emptyset$. (This strict convex cone depends on the choice of norm and need not be the smallest convex cone containing A.) □

We can generalize the existence theorem as follows.

Theorem

Let $\rho : \mathbb{R}^n \to \mathbb{C}$ *be a function with finite support and* ν *a function with discrete support contained in a set* $b + V_\theta^\varphi$, *where* $b \in \operatorname{supp} \nu$, $\theta \in \mathbb{R}^n$, *and* $\varphi \in \Phi$. *Then there is a function* $w : \mathbb{R}^n \to \mathbb{C}$ *with discrete support which solves the equation*

$$\nu * w = \rho.$$

If $w(x) = 0$ *when* $\theta \cdot x \ll 0$, *then* $\operatorname{supp} w \subset -b + V_\theta^\varphi$.

Proof. In view of the normal form it suffices to study the equation $(\delta-\mu)*u=\delta$. Define

$$v=\delta+\mu+\mu*\mu+\mu*\mu*\mu+\cdots=\sum_{j=0}^{\infty}\mu^{*j}.$$

Since we have $\operatorname{supp}(\nu-\nu(b)\delta_b)\subset\{x\in\mathbb{R}^n;\ \theta\cdot x>\theta\cdot b\}$ and the support of ν is discrete, we get $\operatorname{supp}\mu\subset\{x\in\mathbb{R}^n;\ \theta\cdot x\geqslant r\}$ for some positive r. Then

$$\operatorname{supp}\mu^{*j}\subset\{x\in\mathbb{R}^n;\ \theta\cdot x\geqslant jr\},\qquad j\in\mathbb{N},$$

which implies that, in each bounded set, the sum $\sum\mu^{*j}$ is finite.

We have $(\delta - \mu) * v = \delta = (\delta - \mu) * u$. Here $v(x)$ vanishes when $\theta \cdot x < 0$. If also $u(x)$ vanishes when $\theta \cdot x \ll 0$, then it follows that $v = u$.

The support of μ^{*j} is included in $V_\theta^\varphi + V_\theta^\varphi + \cdots + V_\theta^\varphi$ (j terms), which is equal to V_θ^φ in view of our assumption that φ is superadditive. Therefore also the support of v is contained in V_θ^φ. $\square$

Corollary

With ν, b, θ, *and* φ *as in the theorem, let* ρ *now be any function with discrete (possibly infinite) support. Then there is a solution* w *to the equation* $\nu * x = \rho$ *which vanishes for* $\theta \cdot x \ll 0$ *in the following cases.*

1. *If* $\operatorname{supp} \rho \subset c + V_\theta^\varphi$ *for some* c*, then* w *has its support in* $c - b + V_\theta^\varphi$.
2. *If* $\rho * \delta_{-c} \in \mathscr{A}_\theta$ *for some* c*, then* $w * \delta_{b-c} \in \mathscr{A}_\theta$.
3. *If the support of* ρ *is contained in a half space* $\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant s\}$*, then* w *has its support in the half space* $\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant s - \theta \cdot b\}$.

In particular there is a fundamental solution, i.e., a function w such that $\nu * w = \delta$ for certain choices of ν as indicated.

The formula $\sum \mu^{*j}$ lends itself to estimates of the solution. However, it seems to be difficult to get estimates as good as those to be presented later.

The Fourier transformation

We define the Fourier transform $\hat{f}$ of a function $f\colon \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\zeta) = \sum_{x \in \mathbb{R}^n} f(x) e^{i\zeta \cdot x}$$

for those $\zeta \in \mathbb{C}^n$ for which the sum has a good sense. We may take f with support in $\mathbb{Z}^n$, but shall allow also functions defined in $\mathbb{R}^n$ with more general support.

The Fourier transform of a convolution product is given by $(f * g)\hat{} = \hat{f}\,\hat{g}$ under suitable conditions on f and g. The convolution formula $(\delta - \mu) * u = \delta$, where μ has finite support yields

$$\hat{u}(\zeta) = \frac{1}{1 - \hat{\mu}(\zeta)} \qquad \zeta \in \mathbb{C}^n, \quad \operatorname{Im} \zeta_j \text{ large.}$$

We have adapted the signs here to the usual conventions concerning Fourier series. For functions with support in $\mathbb{Z}^n$, the Fourier inversion formula therefore becomes the formula for retrieving the coefficients of the Fourier series, i.e., when $\operatorname{supp} f \subset \mathbb{Z}^n$,

$$f(x) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(\xi) e^{-i\xi\cdot x}\, d\xi_1 \cdots d\xi_n, \qquad x \in \mathbb{R}^n.$$

Here $\xi = (\xi_1, \ldots, \xi_n)$ are n real variables.

However, if the series defining $\hat{f}$ converges well, the formula is valid for functions with arbitrary supports.

Transforms of functions with support in $\mathbb{Z}^n$

For functions with support contained in $\mathbb{Z}^n$ we can profit from the arithmetic of integers to get easy estimates.

If f has its support in $\mathbb{N}^n$ and is of exponential growth, say $|f(x)| \leqslant Ce^{\sigma\cdot x}$, $x \in \mathbb{N}^n$, for some real vector σ, then $\hat{f}$ and $|f|\hat{}$ are well defined and holomorphic in the domain defined by $\operatorname{Im}\zeta_j > \sigma_j$, $j = 1, \ldots, n$, and can be estimated by

$$|\hat{f}(\zeta)| \leqslant ||f|\hat{}(\zeta)| \leqslant C\prod_{j=1}^{n}\frac{1}{1-e^{\sigma_j-\operatorname{Im}\zeta_j}}, \qquad \zeta \in \mathbb{C}^n,\ \operatorname{Im}\zeta_j > \sigma_j.$$

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$$|\hat{f}(\zeta)| \leqslant ||f|\hat{}(\zeta)| \leqslant C \prod_{j=1}^{n} \frac{1}{1 - e^{\sigma_j - \operatorname{Im} \zeta_j}}, \qquad \zeta \in \mathbb{C}^n, \ \operatorname{Im} \zeta_j > \sigma_j.$$

If all the σ_j are negative, the Fourier transform is defined in $\mathbb{R}^n$, but otherwise we have to go out into complex space.

If f has its support in $\mathbb{N}^n$ and grows exponentially, we cannot apply the inversion formula to $\hat{f}$, but to $\hat{f}_\theta$, the Fourier transform of $f_\theta(x) = f(x)e^{\theta\cdot x}$, for a real vector θ satisfying $\theta_j + \sigma_j < 0$, where the σ_j are chosen so that $|f(x)| \leqslant Ce^{\sigma\cdot x}$. We obtain

$$f_\theta(x) = f(x)e^{\theta\cdot x} = (2\pi)^{-n}\int_0^{2\pi}\cdots\int_0^{2\pi}\hat{f}_\theta(\xi)e^{-i\xi\cdot x}\,d\xi_1\cdots d\xi_n,$$

$x \in \mathbb{Z}^n$, where $\hat{f}_\theta(\zeta) = \hat{f}(\zeta - i\theta)$, which means that for $\hat{f}$, the integral goes over a cube in $\mathbb{R}^n$ translated in $\mathbb{C}^n$ by the imaginary vector $-i\theta$.

We can generalize the estimate for $\hat{f}$ to the following.

Lemma

Let $f : \mathbb{Z}^n \to \mathbb{C}$ have support in a cone K and satisfy an estimate $|f(x)| \leqslant Ce^{\sigma \cdot x}$ for $x \in K$. Then the Fourier transform $\hat{f}(\zeta)$ is holomorphic for $\operatorname{Im} \zeta$ in the interior of $\sigma + K^{\text{dual}}$, where K^{dual} is the dual of K, defined as

$$K^{\text{dual}} = \{\eta \in \mathbb{R}^n;\ \eta \cdot x \geqslant 0 \text{ for all } x \in K\}.$$

Proof. By considering $f(x)e^{-\sigma\cdot x}/C$ we are reduced to the case $\sigma = 0$, $C = 1$. We shall thus prove that, if $\|f\|_\infty \leqslant 1$, then $\hat{f}$ is well defined and holomorphic in $\mathbb{R}^n \times i\Lambda$, where Λ is the interior of K^{dual}. We have

$$|\hat{f}(\zeta)| \leqslant \sum_{x\in K\cap\mathbb{Z}^n} e^{-\eta\cdot x}, \qquad \zeta \in \mathbb{C}^n,\ \eta = \operatorname{Im}\zeta.$$

Define cones

$$\Lambda_\tau = \{\eta \in \mathbb{R}^n;\ \eta\cdot x \geqslant \tau\|\eta\|\,\|x\|_1 \text{ for all } x \in K\}, \qquad \tau > 0.$$

The union of all the $\Lambda_\tau \smallsetminus B_<(0,\rho)$, $\tau > 0$, $\rho > 0$, is equal to Λ. Fix τ and ρ. Then, for $\operatorname{Im}\zeta = \eta \in \Lambda_\tau$, we obtain

$$|\hat{f}(\zeta)| \leqslant \sum_{x \in K \cap \mathbb{Z}^n} e^{-\tau\|\eta\|\,\|x\|_1} \leqslant \sum_{x \in \mathbb{Z}^n} e^{-\tau\|\eta\|\,\|x\|_1} = \sum_{x \in \mathbb{Z}^n} \prod_{j=1}^{n} e^{-\tau\|\eta\|\,|x_j|}.$$

When $\tau, \rho > 0$ and $\|\eta\| \geqslant \rho$, the last expression is equal to

$$\prod_{j=1}^{n} \frac{1 + e^{-\tau\|\eta\|}}{1 - e^{-\tau\|\eta\|}} \leqslant \prod_{j=1}^{n} \frac{1 + e^{-\tau\rho}}{1 - e^{-\tau\rho}} < +\infty.$$

Thus $\hat{f}$ is bounded in $\mathbb{R}^n + i(\Lambda_\tau \smallsetminus B_<(0,\rho))$ for every positive τ and ρ, and holomorphic in the interior, hence also holomorphic in the union $\mathbb{R}^n + i\Lambda$ as claimed. $\square$

Theorem

Given a strict closed convex cone K, assume that a function $f\colon \mathbb{Z}^n \to \mathbb{C}$ with support in K satisfies a family of estimates

$$|f(x)| \leqslant Ce^{\sigma \cdot x}, \qquad x \in K,\ \sigma \in \Sigma,$$

for some subset Σ of $\mathbb{R}^n$. Then the Fourier transform of f is holomorphic in the union of all the sets $\mathbb{R}^n + i(\sigma + (K^{\mathrm{dual}})^\circ)$, $\sigma \in \Sigma$.

If f has finite support, or more generally bounded support and if $\|f\|_1$ is finite, its Fourier transform is an entire function. When studying holomorphy of a transform, such functions do not influence the domain. This fact we can use to prove that $\hat{f}$ is holomorphic in a larger domain:

Theorem

Let $f\colon \mathbb{Z}^n \to \mathbb{C}$ be given and define K_r as the smallest closed convex cone containing $\{a \in \operatorname{supp} f;\ \|a\| \geqslant r\}$, $r \geqslant 0$, and K_∞ as the intersection of all the K_r, $0 \leqslant r < +\infty$. Given a vector σ such that, for some cone L such that $L^\circ \supset K_\infty \setminus \{0\}$, $|f(x)| \leqslant Ce^{\sigma \cdot x}$ for all $x \in L$, the Fourier transform $\hat{f}$ of f is holomorphic in

$$\Omega = \mathbb{R}^n + i(\sigma + (K_\infty^{\text{dual}})^\circ).$$

Theorem

Given a function $f\colon \mathbb{Z}^n \to \mathbb{C}$, *define* K_r, $0 \leqslant r \leqslant \infty$, *as before and assume that* f *satisfies a family of estimates*

$$|f(x)| \leqslant Ce^{\sigma\cdot x}, \qquad x \in L_\sigma,\ \sigma \in \Sigma,$$

for some subset Σ *of* $\mathbb{R}^n$, *where, for each* $\sigma \in \Sigma$, L_σ *is a cone such that* $L_\sigma^\circ \supset K_\infty \smallsetminus \{0\}$. *Then the Fourier transform of* f *is holomorphic in the union of all the sets* $\mathbb{R}^n + i(\sigma + (K_\infty^{\mathrm{dual}})^\circ)$, $\sigma \in \Sigma$.

The Fourier transform of functions with more general support

By moving points in the support of a function to an integer point nearby with larger norm we can get a result for functions with arbitrary (not necessarily discrete support):

Theorem

Given $f\colon \mathbb{R}^n \to \mathbb{C}$, *define*

$$f_{\mathbb{Z}}(x) = \sum_{a \in C(x)} |f(a)|, \qquad x \in \mathbb{Z}^n,$$

where $C(x)$, $x \in \mathbb{Z}^n$, *is the set of all* $a \in \mathbb{R}^n$ *such that* $\lceil a_j \rceil = x_j$ *when* $x_j > 0$, $\lfloor a_j \rfloor = x_j$ *when* $x_j < 0$, *and* $C(x)$ *is empty when* $x_j = 0$. *If* $f_{\mathbb{Z}}$ *satisfies an estimate* $|f_{\mathbb{Z}}(x)| \leqslant Ce^{\sigma \cdot x}$ *for* $x \in L \cap \mathbb{Z}^n$, *where* L *is the smallest closed convex cone which contains all points* $x \in \mathbb{Z}^n$ *such that* $C(x)$ *is nonempty, then the Fourier transform* $\hat{f}(\zeta)$ *is holomorphic for* $\operatorname{Im} \zeta$ *in the interior of* $\sigma + L^{\mathrm{dual}}$.

So $C(x)$, when nonempty, is a cube with vertex at $x \in (\mathbb{Z} \smallsetminus \{0\}p)^n$ and such that $\|a\| \leqslant \|x\|$ for all $a \in C(x)$.

So $C(x)$, when nonempty, is a cube with vertex at $x \in (\mathbb{Z} \setminus \{0\}p)^n$ and such that $\|a\| \leqslant \|x\|$ for all $a \in C(x)$.

The cone L spanned by the cubes $C(x)$ can be large, since when $\|x\|$ is small, the cube subtends a big angle as viewed from the origin. But by removing points near the origin we can again get a larger domain of holomorphy:

Theorem

Given a function $f\colon \mathbb{R}^n \to \mathbb{C}$ *such that*

$$\sum_{\substack{a\in\mathbb{R}^n\\ \|a\|<r}} |f(a)|$$

is finite for every r, *define*

$$f_{\mathbb{Z},r}(x) = \sum_{a\in C(x)} |f(a)|, \qquad x\in\mathbb{Z}^n,$$

where for $\|x\| \geqslant r$, $C(x)$ *is defined before, while* $C(x)=\emptyset$ *when* $\|x\|<r$. *If* $f_{\mathbb{Z},r}$ *satisfies an estimate* $|f_{\mathbb{Z},r}(x)| \leqslant Ce^{\sigma\cdot x}$ *for* $x\in L\cap\mathbb{Z}^n$, *where* L *is a closed convex cone such that* L° *contains* $K_\infty \smallsetminus \{0\}$, *then the Fourier transform* $\hat{f}(\zeta)$ *is holomorphic for* $\mathrm{Im}\,\zeta$ *in the interior of* $\sigma + (K_\infty)^{\mathrm{dual}}$.

We can also use an estimate for f itself if its support is sufficiently sparse, and we also here can remove points near the origin.

Theorem

Assume that the support A of a function $f : \mathbb{R}^n \to \mathbb{C}$ is sparse in the sense that $C(x)$, $x \in \mathbb{Z}^n$ contains a number of points in A which grows slower than every exponential function $e^{\varepsilon\|x\|}$, $\varepsilon > 0$. Define K_r, $0 \leqslant r \leqslant \infty$, as before. If f satisfies an estimate $|f(x)| \leqslant Ce^{\sigma \cdot x}$ for $x \in L$, where the interior of L contains $K_\infty \smallsetminus \{0\}$, then the Fourier transform $\hat{f}(\zeta)$ is holomorphic for $\operatorname{Im}\zeta$ in the interior of $\sigma + K_\infty^{\mathrm{dual}}$.

Infimal convolution

Tropicalization means, roughly speaking, that we replace an integral or a sum by a supremum.

A typical example is the tropicalization of the l^p-norm:

$$\|x\|_p = \left(\sum |x_j|^p\right)^{1/p} \text{ is replaced by } \left(\sup |x_j|^p\right)^{1/p} = \sup |x_j| = \|x\|_\infty.$$

In this case we have convergence:

$$\|x\|_p = \left(\sum |x_j|^p\right)^{1/p} \to \sup_j |x_j| = \|x\|_\infty \quad \text{as} \quad p \to +\infty, \ x \in \mathbb{R}^n.$$

Let us study the convolution product of two functions of the form e^{-f}:

$$e^{-h_1(x)} = \sum_{y \in \mathbb{R}^n} e^{-f(x-y)} e^{-g(y)}, \qquad x \in \mathbb{R}^n,$$

assuming that f, g are equal to $+\infty$ outside some discrete set. If for instance f, g have their support in $\mathbb{Z}^n$ and $f(x), g(x) \geqslant \varepsilon \|x\| - C$, we have good convergence: Case ζ'. The tropicalization of this convolution product is

$$e^{-h_\infty(x)} = \sup_{y \in \mathbb{R}^n} e^{-f(x-y)} e^{-g(y)}, \qquad x \in \mathbb{R}^n,$$

which can be written

$$h_\infty(x) = \inf_{y \in \mathbb{R}^n} (f(x-y) + g(y)), \qquad x \in \mathbb{R}^n.$$

Also in this case we have a nice convergence: If we define h_p by

$$e^{-ph_p(x)} = \sum_{y \in \mathbb{R}^n} e^{-pf(x-y)} e^{-pg(y)}, \qquad x \in \mathbb{R}^n,\ p > 0,$$

then h_p converges to h_∞ as $p \to +\infty$.

The function h_∞ is the ***infimal convolution*** of f and g, denoted by $f \sqcap g$. Here we of course need not assume that f and g are equal to $+\infty$ outside a discrete set. More generally, we define it when f and g take values in $\mathbb{R}_!$ using upper addition:

$$(f \sqcap g)(x) = \inf_{y \in \mathbb{R}^n} (f(x-y) \dot{+} g(y)), \qquad x \in \mathbb{R}^n.$$

The function $\mathbf{ind}_{\{0\}}$ is a neutral element for $\sqcap$: $f \sqcap \mathbf{ind}_{\{0\}} = f$ for all f.

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Of course we have also the ***supremal convolution*** defined as

$$(f \sqcup g)(x) = \sup_{y \in \mathbb{R}^n} (f(x-y) \underset{\cdot}{+} g(y)), \qquad x \in \mathbb{R}^n.$$

The superadditive functions in the class Φ can for example be described by the inequality $\varphi \sqcup \varphi \leqslant \varphi$, understanding that $\varphi(t) = -\infty$ for $t < 0$.

The Fenchel transformation

The ***Fenchel transform*** of a function $f\colon \mathbb{R}^n \to \mathbb{R}_!$ is defined as

$$\tilde{f}(\xi) = \sup_{x \in \mathbb{R}^n} (\xi \cdot x - f(x)), \qquad \xi \in \mathbb{R}^n.$$

Clearly $\xi \cdot x - f(x) \leqslant \tilde{f}(\xi)$, which can be written as

$$\xi \cdot x \leqslant f(x) \dot{+} \tilde{f}(\xi), \qquad (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n,$$

called ***Fenchel's inequality***. It follows that the second transform $\tilde{\tilde{f}}$ satisfies $\tilde{\tilde{f}} \leqslant f$. We have equality here if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is $-\infty$ everywhere.

If f is only defined on the integer points, we extend it as $+\infty$ on $\mathbb{R}^n \setminus \mathbb{Z}^n$.

The Fenchel transformation $f \mapsto \tilde{f}$, named for Werner Fenchel (1905–1988), is a tropical counterpart of the Fourier transformation. This is perhaps even more obvious if we look at the Laplace transform of a function g: $(\mathscr{L}g)(\xi) = \int_0^\infty g(x)e^{-\xi x}dx$. If we replace the integral by a supremum and take the logarithm, we get

$$\log(\mathscr{L}_{\text{trop}}\, g)(\xi) = \sup_x(\log g(x) - \xi x) = \tilde{f}(-\xi), \qquad f(x) = -\log g(x).$$

We have

$$(f \sqcap g)\tilde{} = \tilde{f} \underset{\cdot}{+} \tilde{g} \leqslant \tilde{f} \dot{+} \tilde{g}.$$

If φ and ψ are convex, then $\varphi \dot{+} \psi$ is convex, but not always $\varphi \underset{\cdot}{+} \psi$. However, when $\varphi = \tilde{f}$ and $\psi = \tilde{g}$, this is true: $\tilde{f} \underset{\cdot}{+} \tilde{g}$ is always convex, and is often equal to $\tilde{f} \dot{+} \tilde{g}$. In fact equality holds except for a few special cases.

This formula should be compared with $(f * g)\hat{} = \hat{f}\hat{g}$.

Transforms of indicator functions: supporting functions

We define the ***indicator function*** of a set A as the function $\mathbf{ind}_A$ which takes the value 0 in A and $+\infty$ in its complement; thus $\mathbf{ind}_A = -\log \chi_A$, where χ_A is the ***characteristic function*** of A.

If f is an indicator function, then $\tilde{f}$ is positively homogeneous of degree 1 as the supremum of a family of linear functions:

$$\tilde{f}(\xi) = \sup_{\substack{x \in \mathbb{R}^n \\ f(x)=0}} \xi \cdot x, \qquad \xi \in \mathbb{R}^n.$$

Thus the Fenchel transform of an indicator function $\mathbf{ind}_A$ is positively homogeneous of degree 1, and actually equal to the ***supporting function*** H_A of A, which is defined as

$$H_A(\xi) = (\mathbf{ind}_A)^{\tilde{}}(\xi) = \sup_{x \in A} \xi \cdot x, \qquad \xi \in \mathbb{R}^n.$$

Transforms of positively homogeneous functions

Conversely, if φ is positively homogeneous of degree one, then $\tilde{\varphi}$ can take only the values $0, +\infty, -\infty$. Indeed, if $\varphi(tx) = t\varphi(x)$ for all $t > 0$, then $t\tilde{\varphi} = \tilde{\varphi}$ for all $t > 0$, and this is only true for the three values $0, +\infty, -\infty$. The value $-\infty$ will not occur for the functions we are studying, so then $\tilde{\varphi}$ is an indicator function, $\tilde{\varphi} = \mathbf{ind}_M$ for some set M.

If A is a set such that $H_A = (\mathbf{ind}_A)^{\sim} = \varphi$, then $\tilde{\varphi}$ is the second Fenchel transform of $\mathbf{ind}_A$, equal to the indicator function of the closure $\overline{\mathbf{cvxh}(A)}$ of the convex hull $\mathbf{cvxh}(A)$ of A.

Transforms of indicator functions that are positively homogeneous

Let $f = \mathbf{ind}_C$ be an indicator function which is also positively homogeneous. Then C is a cone, and the Fenchel transform of f is also both positively homogeneous and an indicator function, say $\tilde{f} = \mathbf{ind}_\Gamma$, where Γ is a cone, necessarily closed and convex since $\tilde{f}$ is lower semicontinuous and convex.

The dual of a cone C is a closed convex cone: $C^{\text{dual}} = -\Gamma$, where $\mathbf{ind}_\Gamma = (\mathbf{ind}_C)^{\sim}$.

Measuring the growth: The radial indicators

Definition

Given any subset A of $\mathbb{R}^n$ we define A_∞ as the union of $\{0\}$ and the set of all $x \in \mathbb{R}^n \setminus \{0\}$ such that there exists a sequence $(a^{(j)})_j$ of points in A with $\|a^{(j)}\|$ tending to $+\infty$ and $a^{(j)}/\|a^{(j)}\| \to x/\|x\|$.

Measuring the growth: The radial indicators

Definition

Given any subset A of $\mathbb{R}^n$ we define A_∞ as the union of $\{0\}$ and the set of all $x \in \mathbb{R}^n \setminus \{0\}$ such that there exists a sequence $(a^{(j)})_j$ of points in A with $\|a^{(j)}\|$ tending to $+\infty$ and $a^{(j)}/\|a^{(j)}\| \to x/\|x\|$.

If $A = \mathbb{Z}^n$, then $A_\infty = \mathbb{R}^n$. The same holds for all asymptotically dense sets A.

Definition

Given a function $f\colon A \to \mathbb{C}$ we define its ***upper radial indicator*** as

$$p_f(x) = \limsup_{a} \frac{\|x\|}{\|a\|} \log|f(a)|, \qquad x \in A_\infty \setminus \{0\},$$

where the limit superior is taken over all $a \in A$ such that $\|a\| \to +\infty$ and $a/\|a\| \to x/\|x\|$. We define its ***lower radial indicator*** as

$$q_f(x) = \liminf_{a} \frac{\|x\|}{\|a\|} \log|f(a)|, \qquad x \in A_\infty \setminus \{0\}.$$

Finally, we define $p_f(0) = q_f(0) = 0$.

Proposition

Let $f\colon A \to \mathbb{C}$ *be any function and let* $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$. *Then the following four properties are equivalent.*

(A). *For every positive* ε *there exists a constant* C_ε *such that*

$$|f(x)| \leqslant C_\varepsilon e^{\sigma\cdot x + \varepsilon\|x\|}, \qquad x \in A.$$

(A′). *The upper radial indicator of* f *satisfies*

$$p_f(x) \leqslant \sigma \cdot x \qquad x \in A_\infty.$$

(A″). *The Fenchel transform of* $-p_f$ *satisfies* $(-p_f)^{\sim}(-\sigma) \leqslant 0$.

(A‴). $-\sigma \in M_f$, *where* M_f *is the set such that* $(-p_f)^{\sim} = \mathbf{ind}_{M_f}$.

Proposition

If $\mu \colon \mathbb{R}^n \to \mathbb{C}$ *has finite support, then*

$$p_{\hat{\mu}}(\zeta) = q_{\hat{\mu}}(\zeta) = H_{\operatorname{supp}\mu}(-\operatorname{Im}\zeta), \qquad \zeta \in \mathbb{C}^n,$$

the supporting function of the support of μ *evaluated at* $-\operatorname{Im}\zeta$*.*
In particular

$$p_{\hat{\mu}}(-i\eta) = H_{\operatorname{supp}\mu}(\eta), \qquad \eta \in \mathbb{R}^n.$$

Thus both radial indicators are equal and depend only on the convex hull of the support of the function.

Estimates for solutions to convolution equations

Theorem

Let $\mu\colon \mathbb{R}^n \to \mathbb{C}$ *be a nonzero function with finite support and* $\theta \neq 0$ *a given vector in* $\mathbb{R}^n$*. Define*

$$r = \inf_y(\theta \cdot y;\ y \in \operatorname{supp}\mu), \quad R = \sup_y(\theta \cdot y;\ y \in \operatorname{supp}\mu).$$

Assume that r *is positive. Let a real vector* $\sigma = (\sigma_1, \ldots, \sigma_n)$ *be given and define a real number* γ *by*

$$|\mu|\hat{}(i\sigma) = \sum_y |\mu(y)| e^{-\sigma\cdot y} = e^\gamma.$$

Then the unique function $u\colon \mathbb{R}^n \to \mathbb{R}$ *which solves* $(\delta - \mu) * u = \delta$ *and is zero where* $\theta \cdot x \ll 0$*, can be estimated as*

$$|u(x)| \leqslant e^{(\sigma+\gamma\theta/r)\cdot x}, \qquad x \in \mathbb{R}^n, \text{ if } \gamma \geqslant 0; \text{ and}$$

$$|u(x)| \leqslant e^{(\sigma+\gamma\theta/R)\cdot x}, \qquad x \in \mathbb{R}^n, \text{ if } \gamma \leqslant 0.$$

Proof. If $u = \delta + u * \mu$, we have $u(0) = 1$ and, for $x \neq 0$,

$$|u(x)| \leqslant |(u * \mu)(x)| \leqslant \sum_y |\mu(y)|\,|u(x-y)|.$$

If μ is nonnegative and u satisfies the inequality $u \leqslant \delta + u * \mu$, we have $u(0) \leqslant 1$ and the same inequality holds.

Let us try to prove that $|u(x)| \leqslant e^{(\sigma+t\theta)\cdot x}$, where t is a real number to be determined later. Now the values of y for which $\mu(y) \neq 0$ must satisfy $\theta \cdot y \geqslant r$, so that $\theta \cdot (x-y) \leqslant \theta \cdot x - r$. By induction on $\theta \cdot x$ we may therefore assume that all the values of $u(x-y)$ that occur satisfy the estimate. We get

$$|u(x)| \leqslant \sum_y |\mu(y)| e^{(\sigma+t\theta)\cdot(x-y)} \leqslant e^{(\sigma+t\theta)\cdot x} \sum_y |\mu(y)| e^{-(\sigma+t\theta)\cdot y}$$
$$\leqslant \sum_y |\mu(y)| e^{-\sigma\cdot y} \sup_y e^{(\sigma+t\theta)\cdot x} e^{-t\theta\cdot y} \leqslant e^{(\sigma+t\theta)\cdot x} e^{\gamma} \sup_y e^{-t\theta\cdot y}.$$

For $t \geqslant 0$ we have $e^{\gamma} \sup_y e^{-t\theta\cdot y} = e^{\gamma - tr}$; for $t \leqslant 0$ we have $e^{\gamma} \sup_y e^{-t\theta\cdot y} = e^{\gamma - tR}$. We now choose $t = \gamma/r$ if $\gamma \geqslant 0$ and $t = \gamma/R$ if $\gamma \leqslant 0$. This proves the estimate. □

When $\sigma = s\theta$, there is an inverse relation between s and γ: the larger s is, the smaller is γ. It is therefore natural to ask which is the best estimate that can be obtained by this method. The answer is an easy one:

Corollary

Let μ, θ, r and R be as in the theorem, take $\sigma = s\theta$, and define s_0 as the unique real number such that $|\mu|\hat{}(is_0\theta) = 1$. Then the best estimate is obtained when $\gamma = 0$, viz.

$$|u(x)| \leqslant e^{s_0\theta\cdot x}, \qquad x \in \mathbb{R}^n.$$

Example

For the array b of binomial coefficients define as

$$b(x) = \frac{(x_1 + x_2)!}{x_1!x_2!}, \qquad x \in \mathbb{N}^2,$$

as well as for the array b_n of multinomial coefficients

$$b(x) = \frac{(x_1 + x_2 + \cdots + x_n)!}{x_1!x_2!\cdots x_n!}, \qquad x \in \mathbb{N}^n,$$

we choose a θ with all components θ_j positive, and obtain $r = \min_j \theta_j$, $R = \max_j \theta_j$. If $\theta = (1, 1, \ldots, 1)$, then $R = r = 1$, and in general we get $R = r$ if the support of μ is contained in a hyperplane $\{x;\ \theta \cdot x = r\}$, $r > 0$. We have $e^\gamma = \sum_j e^{-\sigma_j}$. If $\sigma = s(1, 1, \ldots, 1)$, we get $s_0 = \log n$, so that $b_n(x) \leqslant n^{x_1 + \cdots + x_n}$, which is the best possible estimate of the form $b_n(x) \leqslant c^{x_1 + \cdots + x_n}$. □

Example

For the array of Delannoy numbers $(d(x,y))_{(x,y)\in\mathbb{Z}^2}$, defined as zero when $x \leqslant -1$ or $y \leqslant -1$, as 1 when $(x,y) = (0,0)$, and for $(x,y) \in \mathbb{N}^2 \smallsetminus \{(0,0)\}$ by the recursion formula

$$d(x,y) = d(x-1,y) + d(x-1,y-1) + d(x,y-1),$$

we have $\|\mu\|_1 = 3$; $\hat{\mu}(i\sigma) = e^{-\sigma_1} + e^{-\sigma_2} + e^{-\sigma_1-\sigma_2} = e^{\gamma}$. For a vector θ with positive components, we have $r = \min(\theta_1, \theta_2)$, $R = \theta_1 + \theta_2$, thus $R \geqslant 2r$. We may take $\theta = (1,1)$, so that $r = 1$ and $R = 2$. Then $\hat{\mu}(is\theta) = 2e^{-s} + e^{-2s}$. Thus for $\sigma = s\theta$, we have $\gamma \geqslant 0$ if and only if $2e^{-s} + e^{-2s} \geqslant 1$, and $\gamma \leqslant 0$ if and only if $2e^{-s} + e^{-2s} \leqslant 1$. The number s_0 is equal to $s = \log(\sqrt{2}+1)$; thus $d(x) \leqslant (\sqrt{2}+1)^{x_1+x_2}$, which is the best possible estimate of the form $d(x) \leqslant c^{x_1+x_2}$ for the Delannoy numbers. $\square$

Conversely we have, under the extra assumption that μ is nonnegative:

Theorem

Let $\mu\colon \mathbb{R}^n \to [0, +\infty[$ *have finite support contained in a half plane* $\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant r\}$, $r > 0$, *and let* K *be the smallest convex cone which contains* $\operatorname{supp}\mu$. *If for any positive* ε *an estimate*

$$u(x) \leqslant C_\varepsilon e^{(\sigma+\varepsilon\theta)\cdot x}, \qquad x \in K,$$

holds for some constant C_ε, *then* $\hat{\mu}(i\sigma) \leqslant 1$.

Proof. We note that $u \geqslant 0$ here since μ is nonnegative.

It is enough to consider the case $\sigma = 0$. Assume that $\|\mu\|_1 > 1$:

$$\|\mu\|_1 = \hat{\mu}(0) = \sum_y \mu(y) > 1.$$

Then $\hat{\mu}(it\theta) = \sum_y \mu(y) e^{-t\theta \cdot y}$, $t \geqslant 0$, takes a value larger than 1 for $t = 0$ and tends to zero when t tends to $+\infty$ since $\theta \cdot y \geqslant r > 0$ in the support of μ. We first determine a positive number s such that $\sum_y \mu(y) e^{-s\theta \cdot y} = 1$. Hence $\hat{\mu}(it\theta)$ is smaller than 1 for $t > s$ and equal to 1 when $t = s$. This implies that

$$\hat{u}(it\theta) = \frac{1}{1 - \hat{\mu}(it\theta)}, \qquad t > s,$$

is finite for $t > s$ and tends to $+\infty$ as $t \searrow s$.

For small enough τ, θ belongs to the cone Λ_τ. Fix such a τ. According to an earlier proof, $\hat{u}(\zeta)$ is bounded for every positive ε when $\tau\|\mathrm{Im}\,\zeta\| \geqslant \varepsilon$; in particular, $\hat{u}(it\theta)$ is bounded when $t\tau\|\theta\| \geqslant \varepsilon$. We can choose $\varepsilon = s\tau\|\theta\|$, so that $\hat{u}(it\theta)$ is bounded for all $t \geqslant s$, contradicting the formula above which shows that $\hat{u}(it\theta)$ tends to $+\infty$ when t tends to s. Hence we cannot have $\hat{\mu}(0) > 1$. $\square$

Theorem

Given a function $\mu\colon \mathbb{R}^n \to [0,+\infty[$ *which is nonzero only at finitely many points in a half plane* $\{x \in \mathbb{R}^n;\ \theta \cdot x \geqslant r\}$, $r > 0$, *let* u *be the unique function* $u\colon \mathbb{R}^n \to \mathbb{C}$ *which is zero where* $\theta \cdot x \ll 0$ *and solves the equation* $(\delta - \mu) * u = \delta$. *Then, given an arbitrary vector* $\sigma \in \mathbb{R}^n$, *the following five conditions are equivalent.*

(A). For every positive ε there exists a constant C_ε such that

$$u(x) \leqslant C_\varepsilon e^{\sigma \cdot x + \varepsilon \|x\|}, \qquad x \in \mathbb{R}^n.$$

(A′). The upper radial indicator of u satisfies

$$p_u(x) \leqslant \sigma \cdot x \qquad x \in \mathbb{R}^n.$$

(A″). The Fenchel transform of $-p_u$ satisfies $(-p_u)^{\sim}(-\sigma) \leqslant 0$.

(B). $u(x) \leqslant e^{\sigma \cdot x}$ for all $x \in \mathbb{R}^n$.

(C). $\hat{\mu}(i\sigma) \leqslant 1$. $\square$

However, I would like to weaken the hypothesis that $\mu \geqslant 0$ ...

We note in particular the implication (A) $\Rightarrow$ (B), which is a kind of Liouville theorem:

Corollary

Let u be as in the theorem. Then $\log u(x) - \varepsilon\|x\|$ is bounded from above for every positive ε if and only if u is bounded. $\square$

Thank you!