Indeterminate moment problems and growth of associated entire functions

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Abstract

To a probability distribution μ with moments of any order

$$s_n = \int x^n \, d\mu(x), \quad n = 0, 1, \ldots, \tag{1}$$

we consider the orthonormal polynomials P_n, i.e.,

$$\int P_n(x) P_m(x) \, d\mu(x) = \delta_{nm}.$$

They satisfy $P^2(z) := \sum |P_n(z)|^2 < \infty$ for all complex z precisely in the indeterminate case, where there are different probability measures with the same moments (1). This leads to a study of entire functions like

$$K(z, w) = \sum_{n=0}^{\infty} P_n(z) P_n(w), \quad z, w \in \mathbb{C}$$

and

$$L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}.$$

During the last 20 years there has been a general study of these entire function as well as many concrete examples, often related to q-series. We will give a review of some of these results together with new results about the relation between the growth of P and summability properties of the sequence $(P_n(z))$. The order of the function P is called the order of the moment problem.

It is shown that under suitable conditions on the recurrence coefficients in the three term recurrence relation

$$z P_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z),$$

the order of the moment problem is equal to the exponent of convergence of the sequence (b_n). Similar results are obtained for logarithmic order and for more general types of slow growth.

The new results are based on joint work with Ryszard Szwarc, Wrocław.