Indeterminate moment problems and growth of associated entire functions

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Based on joint work with
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Main questions

Orthonormal polynomials (P_n) associated to an indeterminate moment problem lead to an entire function

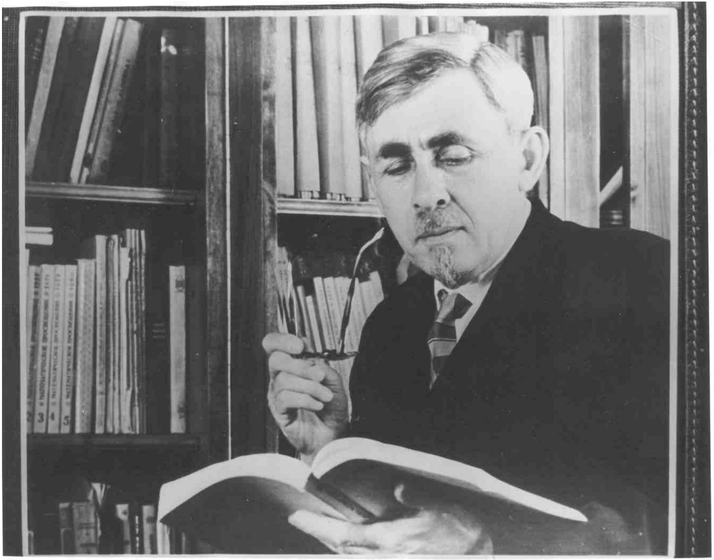
$$D(z) = z\sum_{n=0}^{\infty} P_n(z)P_n(0)$$

of minimal exponential type.
The polynomials are determined by a three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1}P_{n-1}(x), \quad n \geq 0,$$

for certain sequences $a_n \in \mathbb{R}, b_n > 0, n \geq 0$.
The main question to be discussed is recent joint work with Ryszard Szwarc:
Determine the growth properties of D and similar functions in terms of properties of the sequences $(a_n), (b_n)$

My mathematical hero: N.I. Akhiezer (1901-1980)



Overview

1. Introduction and background

   • Introduction to indeterminate moment problems

   • Growth properties of functions: order and type; logarithmic order and type; double logarithmic order and type

   • The order, logarithmic order, double logarithmic order of moment problems

   • How can these numbers be determined from the three term recurrence or the moments?

2. Results about Livšic's function

3. Order functions and their duals

Moment problems

We consider non-degenerate probability measures μ on $\mathbb{R}$ such that $\mathbb{C}[x] \subset L^1(\mu)$. Non-degenerate means that $\mathrm{supp}(\mu)$ is an infinite set.

The corresponding moment sequence is

$$s_n = \int x^n \, d\mu(x), \quad n = 0, 1, \ldots .$$

By a famous result of Hamburger, the sequences arising in this way are characterized by all the Hankel matrices

$$\mathcal{H}_n = \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}, \quad n = 0, 1, \ldots$$

being positive definite (and $s_0 = 1$).

Determinacy/indeterminacy

A Hamburger moment sequence (s_n) can be

1. determinate: Precisely one measure with these moments
2. indeterminate: More than one and hence infinitely many measures with these moments.

An important tool: The orthonormal polynomials $P_n(x), n = 0, 1, \ldots$

$$\int P_n(x) P_m(x)\, d\mu(x) = \delta_{nm}$$

They can be calculated from the moments s_n via the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1} D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad D_n = \det \mathcal{H}_n.$$

Characterization of indeterminacy

The following conditions are equivalent:

(i) Indeterminacy

(ii) $\sum_0^\infty |P_n(i)|^2 < \infty$

(iii) $P^2(z) := \sum_0^\infty |P_n(z)|^2 < \infty$ for all $z \in \mathbb{C}$.

In (iii) the series converges uniformly on compact subsets of $\mathbb{C}$.
The moment problems corresponding to the classical orthogonal polynomial systems: Hermite, Laguerre, Jacobi, Legendre, Chebyshev are determinate.
Stieltjes (1894) gave the first examples of indeterminate measures, e.g. the lognormal distribution in statistics. The polynomials are called Stieltjes-Wigert polynomials.

The log-normal moments

$0 < q < 1$: log-normal moments are $s_n = q^{-n(n+2)/2}$ given by

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)\, dx.$$

Defining

$$h(x) = \sin(\frac{2\pi}{\log(1/q)} \log x)$$

then the non-negative densities $(-1 \le r \le 1)$

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) [1 + rh(x)]$$

and the discrete measures $(a > 0)$

$$\frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{k(k+2)/2} \delta_{aq^k}$$

all have the log-normal moments.

Correspondence between Stieltjes and Hermite

Stieltjes to Hermite: January 30, 1892

"L'existence de ces fonctions $\varphi(x)$ qui, sans être nulles, sont telles que

$$\int_0^\infty x^n \varphi(x)\, dx = 0, \quad n = 0, 1, \ldots,$$

me paraît très remarquable"

$$\varphi(x) = \sin\left(\frac{2\pi}{\log(1/q)} \log x\right) \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)$$

is one of these functions.

The Stieltjes-Wigert polynomials

The orthonormal polynomials are

$$P_n(x;q) = (-1)^n \frac{q^{\frac{n}{2}}}{\sqrt{(q;q)_n}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2+\frac{k}{2}} x^k.$$

Here we have used the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

involving the q-shifted factorial

$$(z;q)_n = \prod_{k=1}^{n} (1 - zq^{k-1}), \quad z \in \mathbb{C}, n = 0, 1, \ldots, \infty.$$

Three-term recurrence relation

The orthonormal polynomials satisfy a second order difference equation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), \quad n \geq 0, \quad \text{(Diff)}$$

for certain sequences $a_n \in \mathbb{R}, b_n > 0, n \geq 0$.
Conversely–Favard's Theorem–given two sequences $a_n \in \mathbb{R}, b_n > 0, n \geq 0$, the initial conditions $P_{-1} = 0, P_0 = 1$ and the difference equation uniquely determine polynomials P_n of degree n which are orthonormal with respect to some probability measure μ as discussed before.
There is a linearly independent solution $(Q_n(x))$ to (Diff) given by

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y}\, d\mu(y).$$

$Q_n(x)$ is a polynomial of degree $n-1$. $Q_{-1} = -1, Q_0 = 0$.

Jacobi matrices

Matrix representation of the operator of multiplication $p(x) \mapsto xp(x)$ in the basis (P_n):

$$J = \begin{pmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This 3-diagonal matrix is called the Jacobi matrix of the moment problem. It acts as a densely defined symmetric operator in the Hilbert space ℓ^2.

The following are equivalent:

- Determinacy $\iff$ defect indices of J equal to $(0,0)$
- Indeterminacy $\iff$ defect indices of J equal to $(1,1)$

Nevanlinna matrix

The following polynomials will be needed

$$\begin{aligned}A_n(z) &= z\sum_{k=0}^{n-1} Q_k(0)Q_k(z),\\ B_n(z) &= -1 + z\sum_{k=0}^{n-1} Q_k(0)P_k(z),\\ C_n(z) &= 1 + z\sum_{k=0}^{n-1} P_k(0)Q_k(z),\\ D_n(z) &= z\sum_{k=0}^{n-1} P_k(0)P_k(z).\end{aligned}$$

In the indeterminate case we can let $n \to \infty$ to get real entire functions A, B, C, D satisfying

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv 1$$

The above matrix is called the Nevanlinna matrix, because Nevanlinna (1922) used it to give the Nevanlinna parametrization of the solutions to the indeterminate moment problem.

Nevanlinna parametrization of solutions ν to $s_n = \int x^n \, d\nu(x), n = 0, 1, \ldots$

The formula

$$\int \frac{d\nu_\varphi(u)}{u - z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

expresses the Stieltjes transform of any solution $\nu = \nu_\varphi$ in terms of a parameter φ running through $\mathcal{P} \cup \{\infty\}$, where $\mathcal{P}$ denotes the set of Pick functions, i.e., the holomorphic functions in the upper half-plane $\mathbb{H}$ with values in $\overline{\mathbb{H}}$.
To any indeterminate moment problem, there are always "many" solution of the following types:

- measures with a C^∞-density
- discrete
- continuous singular

Order of entire functions

For and unbounded continuous function $f : \mathbb{C} \to \mathbb{C}$, the maximum modulus M_f is

$$M_f(r) = \max_{|z| \le r} |f(z)|, \; r \ge 0.$$

The order ρ_f is

$$\rho_f := \inf\{\alpha > 0 \mid M_f(r) \le_{\text{as}} e^{r^\alpha}\},$$

where $\le_{\text{as}}$ means that it holds for r sufficiently large. ($\rho_f = \infty$ if no such α exists.) Clearly

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

$\sin, \exp$ are entire functions of order 1.

$\exp(P(z))$ is of order n, if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots$ is a polynomial of degree n.

$\exp(\exp z)$ is of order ∞.

Type of functions

For functions of order $0 < \rho < \infty$ the type τ_f is

$$\tau_f := \inf\{c > 0 \,|\, M_f(r) \leq_{\text{as}} e^{cr^\rho}\},$$

so the type of sin, exp is 1, $1/\Gamma(z)$ is of type ∞, while $\exp(P(z))$ above has type $|a_n|$. Clearly

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}.$$

A function f is of minimal exponential type if

$$\forall \varepsilon > 0 : \; M_f(r) \leq_{\text{as}} \exp(\varepsilon r).$$

This is equivalent to: Either $\rho_f < 1$ or $\rho_f = 1$ and $\tau_f = 0$.

Work about Nevanlinna matrices

All the functions A, B, C, D are of minimal exponential type (Marcel Riesz (1923)).
The beautiful work of Akhiezer: The classical moment problem (Russian edition 1961, English 1965) contains no examples of indeterminate moment problems with explicit Nevanlinna matrix.
The first complete Nevanlinna matrices were published in 1993-1994:

- Ismail-Masson: The functions are of order 0 related to theta-functions
- Berg-Valent: The functions are of order $1/4$. The simplest one is

$$D(z) = \frac{4}{\pi}\sqrt{z}\sin(kz^{1/4})\sinh(kz^{1/4}),$$

where $k > 0$ is some constant.

Further results about Nevanlinna matrices

- Berg-Pedersen (1994) The functions A, B, C, D, P have the same order and type.
  The order ρ and type τ of these functions are called the order and type of the moment problem.
- Possible pairs (ρ, τ) for indeterminate moment problems are $(1, 0)$ and $]0, 1[\times[0, \infty]$.
- Many indeterminate moment problems occur within the so-called q-Askey scheme of orthogonal polynomials. They were all classified in the thesis of J.S. Christiansen (2004) and they have all order 0.
- Berg-Pedersen (2005) showed using a refined growth scale called logarithmic order and type: The functions A, B, C, D, P from an indeterminate moment problem of order zero have the same logarithmic order and type called the logarithmic order and type of the moment problem.

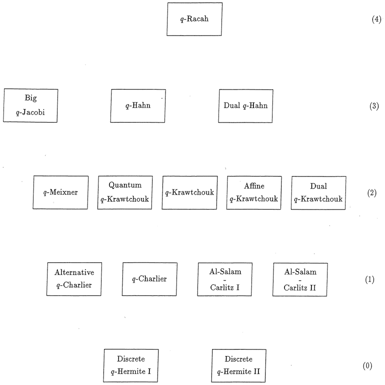
q-Askey scheme

SCHEME I

OF

BASIC HYPERGEOMETRIC

ORTHOGONAL POLYNOMIALS



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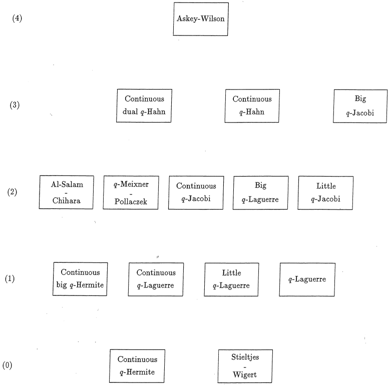
q-Askey scheme

SCHEME II
OF
BASIC HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS



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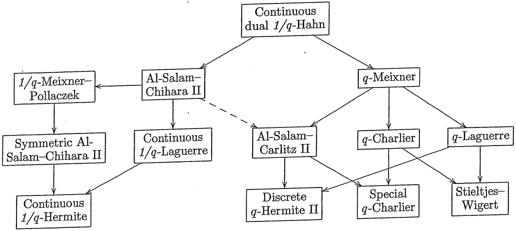

Figure : Christiansen's scheme of indeterminacy

Logarithmic order and type

For an unbounded continuous function f we define the logarithmic order $\rho_f^{[1]}$

$$\rho_f^{[1]} = \inf\{\alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha}\}.$$

Of course $\rho_f^{[1]} < \infty$ is only possible for functions of order 0.
It is easy to obtain that

$$\rho_f^{[1]} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} - 1.$$

When $\rho_f^{[1]} < \infty$ we define the logarithmic type $\tau_f^{[1]}$

$$\tau_f^{[1]} = \inf\{c > 0 \mid M_f(r) \leq_{\text{as}} r^{c(\log r)^{\rho_f^{[1]}}}\},$$

and it is readily found that

$$\tau_f^{[1]} = \limsup_{r\to\infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}+1}}.$$

The D-function for the Stieltjes-Wigert case

$$D(z) = \frac{z}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} (-zq^{3/2})^n$$

so it is closely related to the Rogers-Ramanujan function

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} z^n.$$

It appears in Ramanujan's lost notebook, and the zeros have been studied by Andrews(2005), Bergweiler-Hayman(2003), Hayman(2005), Ismail-Zhang(2007) and Huber(2010). They are not explicitly known.
We have $\rho_\Phi^{[1]} = 1$ and $\tau_\Phi^{[1]} = (4\log(1/q))^{-1}$.

Replacing ℓ^2 by ℓ^α

Theorem (1.B.-Szwarc(2012))

For a moment problem and $0 < \alpha \le 1$ the following conditions are equivalent:

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$,

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$. *Furthermore,* $(1/b_n) \in \ell^\alpha$ *and*

$$P(z) \le C \exp(K|z|^\alpha),$$

where

$$C = \left(\sum_{n=0}^{\infty}(P_n^2(0) + Q_n^2(0))\right)^{1/2}, \; K = \frac{1}{\alpha}\sum_{n=0}^{\infty}(|P_n(0)|^{2\alpha} + |Q_n(0)|^{2\alpha}).$$

The moment problem has order $\rho \le \alpha$*, and if the order is* α*, then the type* $\tau \le K$.

Indeterminacy in terms of a_n, b_n

Indeterminacy implies that $b_n \to \infty$.
More is known:

Carleman's Criterion(1926): Indeterminacy $\Rightarrow \sum 1/b_n < \infty$
Warning: There exist determinate problems with $\sum 1/b_n < \infty$, but this is not possible if (b_n) has a "regular" behaviour.
We start with an elementary Lemma:

Lemma

Let $b_n > 0$ satisfy $\sum 1/b_n < \infty$ and assume that b_n is eventually log-convex (i.e., $b_n^2 \le b_{n-1}b_{n+1}, n \ge n_0$) or eventually log-concave (i.e., $b_n^2 \ge b_{n-1}b_{n+1}, n \ge n_0$), then b_n is eventually strictly increasing to infinity.

Carleman plus "extra condition" implies indeterminacy

Theorem (2.B.-Szwarc(2012))

Assume that the coefficients a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that b_n is either eventually log-convex or eventually log-concave.
The for a constant c independent of z

$$\sqrt{b_{n-1}}|P_n(z)| \le c\,\Pi(|z|), \quad \Pi(z) = \prod_{k=0}^{\infty}\left(1+\frac{z}{b_{k-1}}\right), \quad n \ge 0,$$

The moment problem is indeterminate.

This extends a result of Berezanskiĭ (1956).

Some comments

Under the conditions of the theorem we can further obtain

$$P_n^2(0) = O(1/b_{n-1}); 1/b_n = o(1/n)$$

$$\frac{K}{b_{n+1}} \le |P_n(z)|^2 + |P_{n+1}(z)|^2 \le \frac{L}{b_{n-1}}$$

for suitable constants K,L depending on z (but not on n). Similar results are true for Q_n.

In the symmetric case $a_n = 0$ the sum condition

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

is equivalent to $\sum 1/b_n < \infty$.

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$$\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

is equivalent to $\sum 1/b_n < \infty$.

A partial converse of Theorem 2

Theorem (3.B.-Szwarc(2012))

Assume that a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that logconvexity/logconcavity holds. Assume in addition that P satisfies

$$P(z) \le C \exp(K|z|^{\alpha})$$

for some α such that $0 < \alpha < 1$ and suitable constants $C, K > 0$. Then

$$1/b_n, P_n^2(0), Q_n^2(0) = O(n^{-1/\alpha}),$$

so in particular $(1/b_n), (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$ for any $\varepsilon > 0$.

Summing up

Assume $0 < \alpha < 1$.
Then

$$(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha \Rightarrow P(z) \leq C \exp(K|z|^\alpha)$$

Under logconvexity/logconcavity and the sumcondition

$$P(z) \leq C \exp(K|z|^\alpha) \Rightarrow (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$$

for any $\varepsilon > 0$.

Exponent of convergence for a sequence

For a sequence (z_n) of complex numbers for which $|z_n| \to \infty$, we introduce the exponent of convergence

$$\mathcal{E}(z_n) = \inf \left\{ \alpha > 0 \mid \sum_{n=n^*}^{\infty} \frac{1}{|z_n|^\alpha} < \infty \right\},$$

where $n^* \in \mathbb{N}$ is such that $|z_n| > 0$ for $n \geq n^*$.
The counting function of (z_n) is defined as

$$n(r) = \#\{n \mid |z_n| \leq r\}.$$

The following result is well-known

Lemma

$$\mathcal{E}(z_n) = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.$$

Theorem (4.B.-Szwarc(2012))

Assume that a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that logconvexity/logconcavity holds.
Then the order ρ of the moment problem is given by $\rho = \mathcal{E}(b_n)$.
If $\rho = 0$, then the logarithmic order $\rho^{[1]}$ is given by $\rho^{[1]} = \mathcal{E}(\log b_n)$.

Some examples

1. For $\alpha > 1$ let $b_n = (n+1)^\alpha, a_n = 0, n \ge 0$. The three-term recurrence relation with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying the logconcavity condition and $\sum(1/b_n) < \infty$. By Theorem 4 the order of the moment problem is $1/\alpha$.
2. $b_n = (n+1)\log^\alpha(n+2), a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.
3. For $a > 1, \alpha > 0$ let

$$b_n = a^{n^{1/\alpha}}, \quad |a_n| \le a^{cn^{1/\alpha}} \text{ where } 0 < c < 1.$$

$$b_n^2 \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} b_{n-1}b_{n+1} \Leftrightarrow \left\{ \begin{array}{l} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{array} \right. .$$

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$.

Double logarithmic order and type

4. For $a, b > 1$ let

$$b_n = a^{b^n}, \quad |a_n| \le a^{cb^n} \text{ with } bc < 1.$$

(b_n) is logconvex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0. This motivates a refined growth scale: For an unbounded continuous function f we define the double logarithmic order $\rho_f^{[2]}$ as

$$\rho_f^{[2]} = \inf\{\alpha > 0 \mid M_f(r) \le_{\text{as}} r^{(\log\log r)^\alpha}\}.$$

Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$.
In case $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$ we define the double logarithmic type as

$$\tau_f^{[2]} = \inf\{c > 0 \mid M_f(r) \le_{\text{as}} r^{c(\log\log r)^{\rho^{[2]}}}\}.$$

Double logarithmic order and type of moment problems

Theorem (5.B.-Szwarc(2012))

For an indeterminate moment problem of logarithmic order zero the functions A, B, C, D, P, Q have the same double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ called the double logarithmic order and type of the moment problem.
Under the sum condition and logconvexity/logconcavity

$$\rho^{[2]} = \mathcal{E}(\log \log b_n).$$

An example

Example 5. $b_n = \exp(e^{n^{1/\alpha}})$ is eventually log-convex because $\exp(x^{1/\alpha})$ is convex for $x > (\alpha - 1)^\alpha$ when $\alpha > 1$ and convex for $x > 0$ when $0 < \alpha \leq 1$. The indeterminate moment problem with recurrence coefficients $a_n = 0$ and b_n as above has double logarithmic order equal to $\mathcal{E}(\log \log b_n) = \alpha$.
The function

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\exp(e^{n^{1/\alpha}})} \right),$$

where $0 < \alpha < \infty$, has $\rho_f^{[2]} = \alpha,\ \tau_f^{[2]} = 1$.

Livšic's function

For an indeterminate moment sequence (s_n) Livšic(1939) considered the function

$$L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}.$$

It is entire of minimal exponential type.
Livšic proved that $\rho_L \leq \rho_B$. We know now that $\rho_B = \rho$: the order of the moment problem. It is interesting to know whether the equality sign holds. In fact, we do not know any example with $\rho_L < \rho$. We will also consider the functions

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_n^n}, \quad H(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_{n,n}}.$$

Here $b_{n,n}$ is the leading coefficient of $P_n(z)$.

The good cases where all orders agree

Theorem (6.B.-Szwarc(2012))

Given an (indeterminate) moment problem where

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and where the logconvexity/logconcavity condition holds.
Then

(i) $\rho = \rho_G = \rho_H = \rho_L = \mathcal{E}(b_n)$.
If $\rho = 0$ *then*

(ii) $\rho^{[1]} = \rho_G^{[1]} = \rho_H^{[1]} = \rho_L^{[1]} = \mathcal{E}(\log b_n)$.
If $\rho^{[1]} = 0$ *then*

(iii) $\rho^{[2]} = \rho_G^{[2]} = \rho_H^{[2]} = \rho_L^{[2]} = \mathcal{E}(\log\log b_n)$.

Order functions

Some of the previous results can be generalized using a concept of an order function and its dual function. Some of our proofs also depend on these notions.

Definition. An order function is a continuous, positive and increasing function $\alpha : (r_0, \infty) \to \mathbb{R}$ such that $\lim_{r\to\infty} \alpha(r) = \infty$ and such that the function $r/\alpha(r)$ is also increasing with $\lim_{r\to\infty} r/\alpha(r) = \infty$. Here $0 \le r_0 < \infty$.

If α is an order function, then so is $r/\alpha(r)$.

Definition. For an order function α the function

$$\beta(r) = \frac{1}{\alpha(r^{-1})}, \quad 0 < r < r_0^{-1}$$

will be called the dual function.

Since $\lim_{r\to 0} \beta(r) = 0$, we define $\beta(0) = 0$. Note that β as well as $r/\beta(r)$ are increasing.

Examples of order functions and their dual

1.

$$\alpha(r) = r^\alpha, \quad \beta(r) = r^\alpha, \quad 0 < \alpha < 1, \quad r_0 = 0.$$

2.

$$\alpha(r) = \log^\alpha r, \quad \beta(r) = \frac{1}{(-\log r)^\alpha}, \quad \alpha > 0, \quad r_0 = \exp(\alpha).$$

3.

$$\alpha(r) = \log^\alpha \log r, \quad \beta(r) = \frac{1}{\log^\alpha(-\log r)}, \quad \alpha > 0,$$

where $r_0 > \mathrm{e}$ is the unique solution to $(\log r)\log\log r = \alpha$.

4.

$$\alpha(r) = (\log^\alpha r)\log^\beta \log r, \quad \alpha, \beta > 0.$$

5. If α is an order function, then so are $c\alpha(r)$ and $\alpha(cr)$ for $c > 0$.

6. If α_1 and α_2 are order functions, then $\alpha_1(\alpha_2(r))$ is an order function for r sufficiently large.

Remarks

Let α be an order function with dual function β. If u_n is a sequence of non-negative numbers tending to zero, then $\beta(u_n)$ is only defined for n sufficiently large, but statements like

$$\sum_{n}^{\infty} \beta(u_n) < \infty, \quad \beta(u_n) = O(1/n)$$

make sense.

Definition. We say that a continuous unbounded function $f : \mathbb{C} \to \mathbb{C}$ has order bounded by $\alpha(r)$ if

$$M_f(r) \leq_{\text{as}} r^{K\alpha(r)},$$

for some constant $K > 0$.

Note. To have order bounded by $\alpha(r) = \log^{\alpha}(r)$ for some $\alpha > 0$, is the same as to have finite logarithmic order.

Theorem (7.B.-Szwarc(2012))

For an order function α with dual function β the following conditions are equivalent for a given indeterminate moment problem:

(i) $\beta(P_n^2(0)), \beta(Q_n^2(0)) \in \ell^1$,

(ii) $\beta(|P_n(z)|^2), \beta(|Q_n(z)|^2) \in \ell^1$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, then the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$.
Furthermore, $\beta(1/b_n) \in \ell^1$ *and* P *has order bounded by* α.

Theorem (8.B.-Szwarc(2012))

Assume that a_n, b_n satisfy the sum-condition and the logconvexity/logconcavity condition. Assume in addition that the function $P(z)$ has order bounded by some given order function α.

(i) *If there is $0 < \alpha < 1$ so that $r^\alpha \leq_{as} \alpha(r)$, then*

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) = O\left(\frac{\log n}{n}\right).$$

(ii) *If $\alpha(r^2) = O(\alpha(r))$, then*

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) = O(1/n).$$

In both cases

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) \in \ell^{1+\varepsilon}$$

for any $\varepsilon > 0$.

Thank you for your attention

For more details see

C. Berg and R. Szwarc, *On the order of indeterminate moment problems*, Advances in Mathematics **250** (2014), 105–143.

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