Numerical reconstruction of polytopes from directional moments.

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n-dimensional shape-from-moments problem

Reconstruct a polytope $\mathcal{P}$ from a finite set of its moments.

Retrieve $\mathcal{V}$ the set of vertices of $\mathcal{P}$ from moments.

Previously

- 2D-case solved by computing complex vertices and using numerical methods (Milanfar *et al.* 95, Golub *et al.* 99)
- convex polytopes solved for exact computation using Brion's identities (Gravin,Pasechnik,Lasserre,Robins 12)

Our method

Combine numerical methods of the 2D-case with theoretical results of the higher dimensional case; solve the underlying problems with an algorithm numerically valid.

Plan

Plan

1. Directional moments
   - Real, complex and directional moments
   - Brion's identities
2. Recovering the vertices from directional moments
3. Simulations

Consider a polytope $\mathcal{P}$ in $\mathbb{R}^n$.

Real moments of order k

$$m_{k_1,k_2,\ldots,k_n} = \int_{\mathcal{P}} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \; dx_1 dx_2 \ldots dx_n,$$

with $k_1 + k_2 + \ldots + k_n = k$.

Complex moment of order k

$$m_k(1,i) = \int_{\mathcal{P}} z^k dx_1 dx_2, \text{ where } z = x_1 + i\; x_2.$$

Directional moment of order k

$$m_k(\delta) = \int_{\mathcal{P}} \langle x, \delta \rangle^k dx = \int_{\mathcal{P}} (x_1\delta_1 + \ldots + x_n\delta_n)^k dx_1 dx_2 \ldots dx_n,$$

where δ is the unit vector on the direction.

Brion's identities

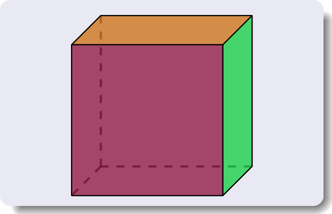
Consider a convex polytope $\mathcal{P}$ in $\mathbb{R}^n$ with r vertices.
Assume that the orthogonal projections on the direction δ of the vertices in $\mathcal{V}$ are distinct. Then

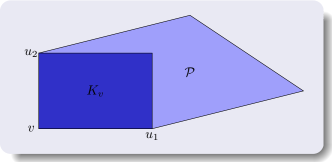
$$\frac{(k+n)!}{k!(-1)^n} m_k(\delta) = \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n+k}, \qquad k \geq 0$$
$$0 = \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{k}, \qquad 0 \leq k \leq n-1,$$

where the coefficients $a_v(\delta)$ depend on δ and the adjacent vertices of v in a triangulation of $\mathcal{P}$.
Moreover

$$a_v(\delta) \neq 0 \quad \forall\ v \in \mathcal{V}.$$

Case of a simple convex polytope $\mathcal{P}$





In this case, the coefficients $a_v(\delta)$ are given by

$$a_v(\delta) = \frac{V_v}{\prod_{u \in K_v} \langle u - v, \delta \rangle},$$

where V_v is the volume of the parallelepiped K_v defined by the edges adjacent to v.

Davis' formula

For any polygon $\mathcal{V}$ in the complex plane and for any analytic function f,

$$\iint_{\mathcal{V}} f''(z)\, dx_1 dx_2 = \sum_{v \in \mathcal{V}} a_v\, f(v),$$

where v are the complex vertices.

In this case, the coefficients a_v - or $a_v(\delta)$ with $\delta = (1, \mathrm{i})$ - are given by

$$a_v(\delta) = \frac{V_v}{\prod_{u \in K_v} \langle u - v, \delta \rangle},$$

where V_v is the volume of the parallelogram K_v defined by the edges adjacent to v.

Plan

Reconstruction of a n-dimensional convex polytope $\mathcal{P}$ with r vertices. First, we assume we know r.

Algorithm

1. recovering the projections $\mathcal{V}(\delta) = \{\langle v, \delta \rangle \mid v \in \mathcal{V}\}$ of the r vertices of $\mathcal{P}$ from its directional moments $m_k(\delta)$.
2. recovering the set of vertices $\mathcal{V}$ from $n+1$ sets $\mathcal{V}(\delta)$.

Key assumption

The projections of the vertices on the chosen directions δ are pairwise distinct.

From the moments to an appropriate sequence

Consider the sequence $(\mu_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} \mu_k = 0 & \text{for } 0 \le k \le n-1 \\ \mu_k = \dfrac{k!(-1)^n}{(k-n)!}\, m_{k-n}(\delta) & \text{for } k \ge n. \end{cases}$$

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Thus we have a sequence $(\mu_k)_{k\in\mathbb{N}}$ such that

$$\mu_k = \sum_{i=1}^{r} a_i\, w_i^k,\ \forall k \in \mathbb{N},$$

with $a_1, \ldots, a_r$ non-zero real numbers and $\mathcal{V}(\delta) = (w_1, \ldots, w_r)$ the sought numbers.

Recurrence equation of order r

Such a sequence is a solution to the recurrence equation of order r

$$\mu_{k+r} = p_{r-1}\,\mu_{k+r-1} + \ldots + p_0\,\mu_k,$$

where $(-p_0, \ldots, -p_{r-1}, 1)$ are the coefficients of the characteristic polynomial

$$p(w) = \prod_{i=1}^{r}(w - w_i) = w^r - p_{r-1}\,w^{r-1} - \ldots - p_1\,w - p_0.$$

Prony's method

Linear system

$$\underbrace{\begin{pmatrix} \mu_0 & \mu_1 & \cdots & & \mu_{r-1} \\ \mu_1 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \mu_{r-1} & & \cdots & & \mu_{2r-2} \end{pmatrix}}_{H_0} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ p_{r-1} \end{pmatrix} = \begin{pmatrix} \mu_r \\ \mu_{r+1} \\ \vdots \\ \vdots \\ \mu_{2r-1} \end{pmatrix}.$$

2-step procedure

1. Solving this linear system gives the coefficients of the polynomial p.
2. Finding the roots of p gives the sought $\mathcal{V}(\delta)$.

Pencil method

Matrix equality

$$H_0 C_p = H_1,$$

where H_0 is the Hankel matrix in Prony's method, H_1 is the shifted Hankel matrix and C_p is the companion matrix of the polynomial p.

1-step procedure

Consider the generalized eigenvalues problem for the pair of matrices (H_1, H_0).

$$H_1 W^{-1} = H_0 W^{-1} D,$$

where D is the diagonal matrix with diagonal elements $\mathcal{V}(\delta)$ and W is the Vandermonde matrix defined by $\mathcal{V}(\delta)$.

Set of solutions

$\mathcal{V}(\delta)$ are the projections $\langle v_j, \delta \rangle$ of the vertices v_j on the direction δ.

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2D-case

Using complex vertices $\delta = (1, i)$,
this is sufficient for reconstructing a convex polygon.

Set of solutions

$\mathcal{V}(\delta)$ are the projections $\langle v_j, \delta \rangle$ of the vertices v_j on the direction δ.

2D-case

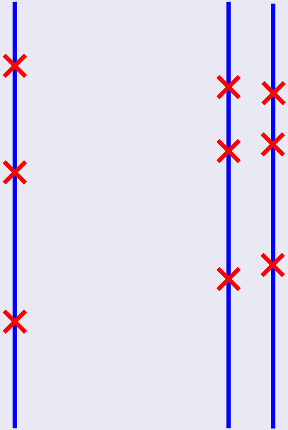
Using complex vertices $\delta = (1, i)$,
this is sufficient for reconstructing a convex polygon.

nD-case

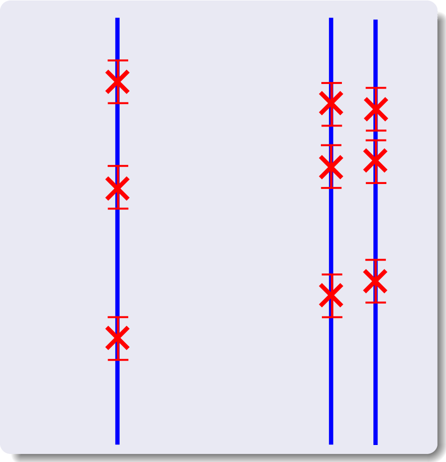
Using real numbers,
this method needs to be repeated for at least n directions.

Interval interpolation

We have $n+1$ sets of projected vertices $\mathcal{V}(\delta)$.



Interval interpolation



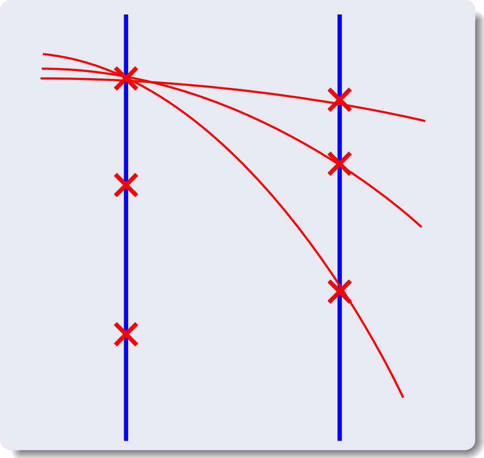
Intervals

$[w - cw\varepsilon\kappa, w + cw\varepsilon\kappa]$,
where $w \in \mathcal{V}(\delta)$,
ε error term
and κ conditioning term.

Conditioning

An upper bound for the conditioning κ of the generalized eigenvalues problem is given by $\kappa(W)^2$. (Beckermann,Golub,Labahn 07)

Interval interpolation

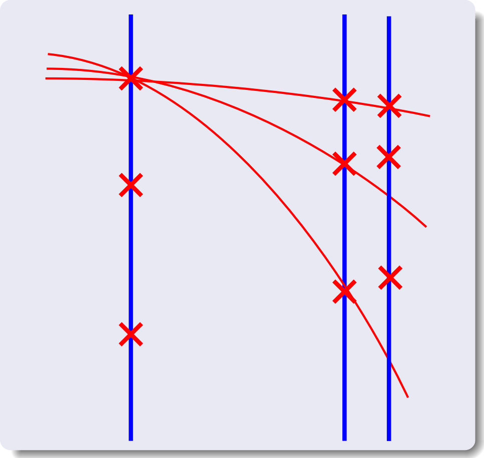


First step

From the n directions with the lowest κ, we construct all possible linear interpolants of the form

$$< v, \delta >= \sum_{i=1}^{n} v_i \delta_i .$$

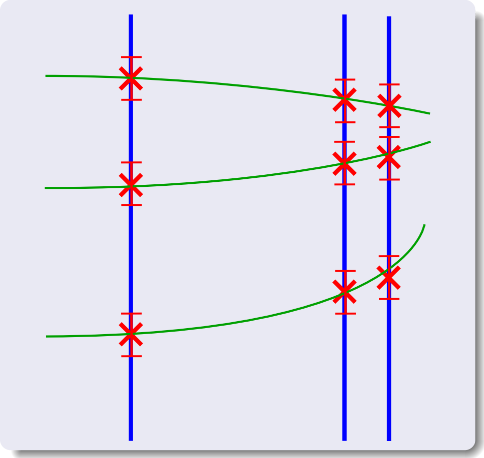
Interval interpolation



Second step

We add the additional direction.

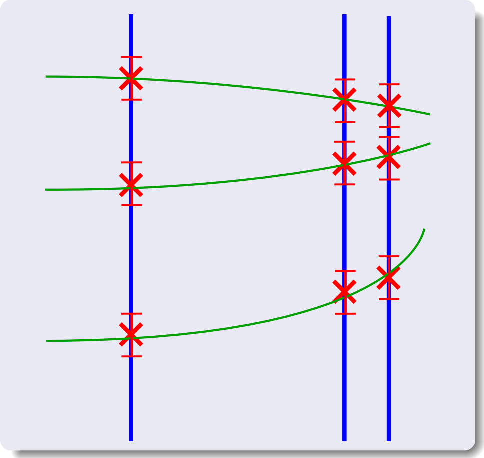
Interval interpolation



Second step

We keep the r interpolants passing through the intervals.

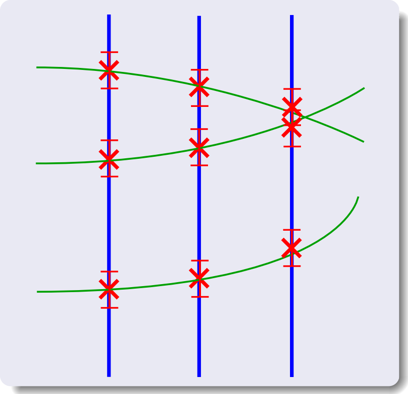
Interval interpolation



Third step

With the help of the intervals, we compute the r best interpolants. (Salazar Celis *et al.* 07)

Intervals problem



Problem

If two intervals are not disjoint, then we cannot match the projections together.

Solutions

1. to choose directions with a better conditioning κ
2. to increase the working precision ε

Reconstruction of a n-dimensional convex polytope $\mathcal{P}$ with r vertices if an upper bound R is known.

Algorithm

1. estimating the number of vertices r.
2. recovering the projections $\mathcal{V}(\delta) = \{\langle v, \delta \rangle \mid v \in \mathcal{V}\}$ of the r vertices of $\mathcal{P}$ from its directional moments $m_k(\delta)$.
3. recovering the set of vertices $\mathcal{V}$ from $n+1$ sets $\mathcal{V}(\delta)$.

Key assumption

The projections of the vertices on the chosen directions δ are pairwise distinct.

Estimating the number of vertices

Factorization of the Hankel matrix H_0 of size $k \times k$, $k \geq 1$

$$H_0 = W^t \; A \; W,$$

where W is the Vandermonde matrix of size $r \times k$ defined by the set of projected vertices $V(\delta)$ and A is the diagonal matrix of size $r \times r$ whose elements are the non-zero coefficients $a_v(\delta)$.

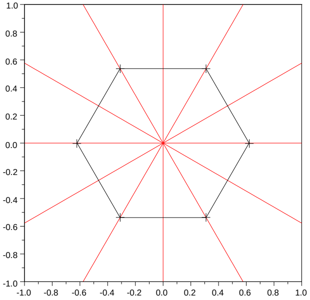
Numerical rank by Singular Values Decomposition

If an upper bound R for r is known,
then analysing the singular values of the Hankel matrix H_0 of size $R \times R$ gives us an estimation of r.

Plan

1. Directional moments
2. Recovering the vertices from directional moments
3. Simulations
   - First example : regular hexagon
   - Second example : polygon with 12 vertices
   - Third example : non-convex polygon
   - Diamond

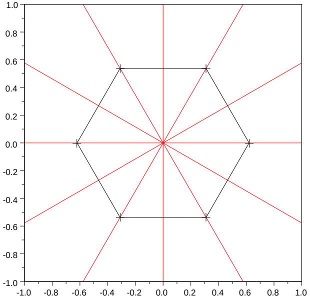
Regular hexagon



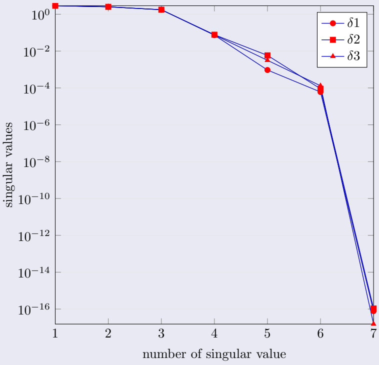
Regular centered-scaled hexagon

x_1-coordinates	x_2-coordinates
0.3102	0.5373
-0.3102	0.5373
-0.6204	0.0000
-0.3102	-0.5373
0.3102	-0.5373
0.6204	0.0000

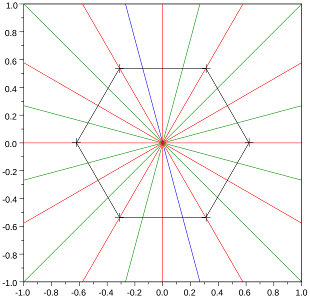
Regular hexagon



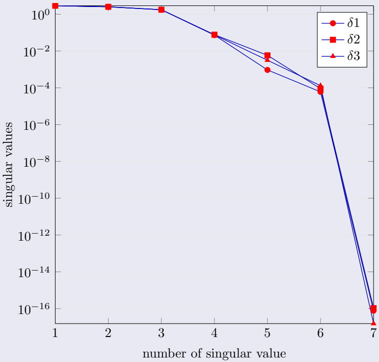
Estimation of the number of vertices



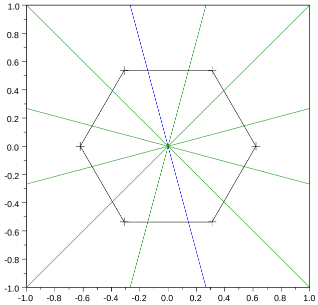
Regular hexagon



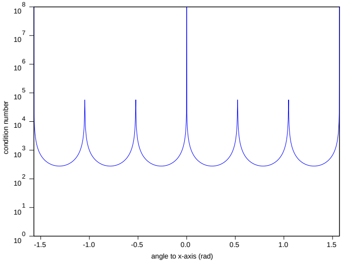
Estimation of the number of vertices



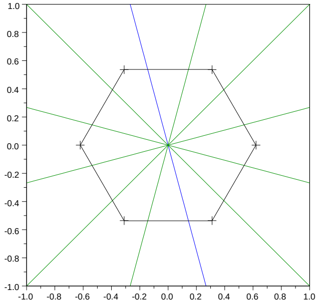
Regular hexagon



Choice of a reference direction



Regular hexagon

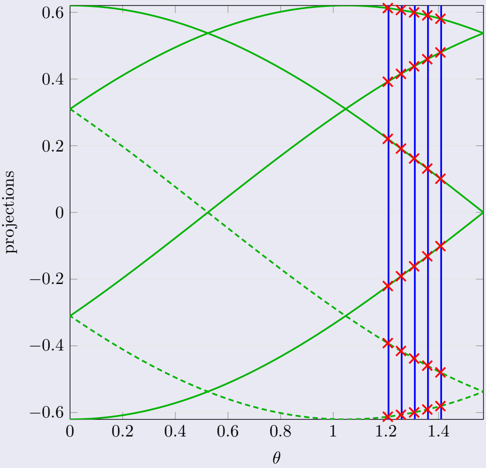


Pencil method for the reference direction

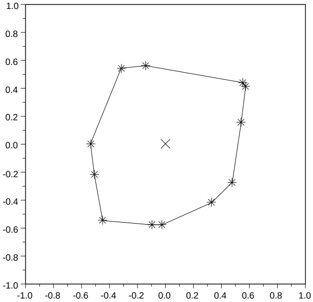
Reference direction		$(\cos\theta, \sin\theta)$
	θ	1.30724
Conditioning		
	$\kappa(W)$	$2.8\ 10^2$
Projections		$\mathcal{V}(\delta)$
0.5995	0.4379	0.1616
-0.1616	-0.4379	-0.5995
Maximum error		
	Err_w	$5.8\ 10^{-14}$

Regular hexagon

Interval interpolation



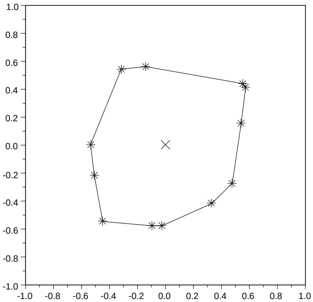
Polygon with 12 vertices



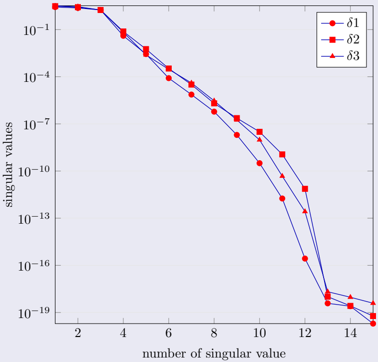
Centered-scaled polygon

x_1-coordinates	x_2-coordinates
-0.0960	-0.5764
-0.0243	-0.5764
0.3309	-0.4170
0.4777	-0.2713
0.5415	0.1578
0.5734	0.4140
0.5506	0.4413
-0.1404	0.5620
-0.3157	0.5449
-0.5354	0.0042
-0.5081	-0.2167
-0.4489	-0.5457

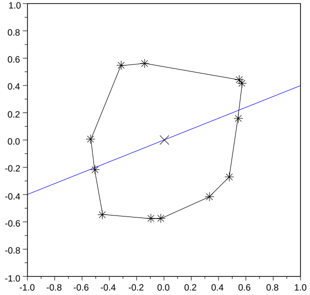
Polygon with 12 vertices



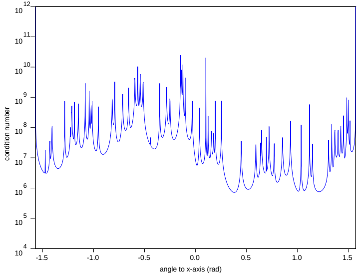
Estimation of the number of vertices



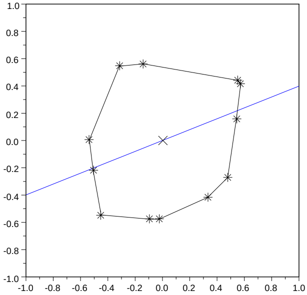
Polygon with 12 vertices



Choice of a reference direction



Polygon with 12 vertices

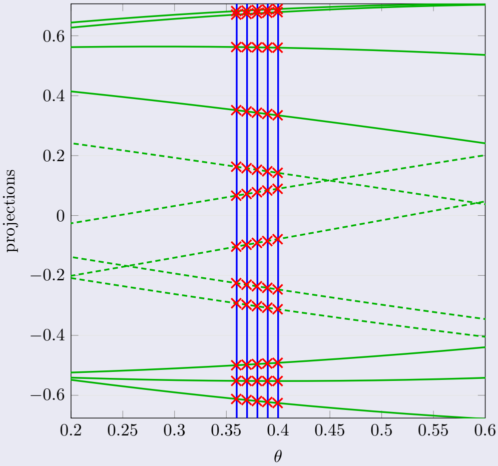


Pencil method for the reference direction

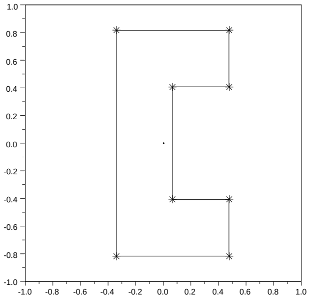
	Reference direction	$(\cos\theta, \sin\theta)$
	θ	0.379521
	Conditioning	
	$\kappa(W)$	$7.2\ 10^5$
	Projections	$\mathcal{V}(\delta)$
0.6859	0.6749	0.5614
0.3432	0.1528	0.0778
−0.0914	−0.2361	−0.3027
−0.4958	−0.5522	−0.6191
	Maximum error	
	Err_w	$2.7\ 10^{-8}$

Polygon with 12 vertices

Interval interpolation



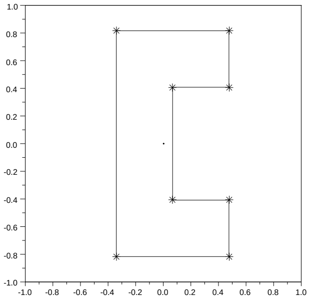
Non-convex polygon : C-shape



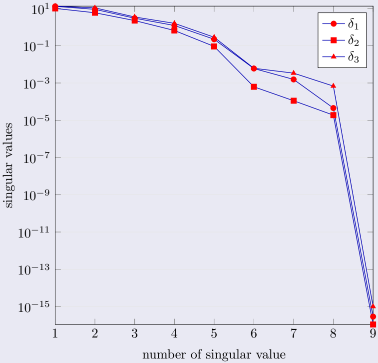
Centered-rescaled polygon

x_1-coordinates	x_2-coordinates
-0.3402	-0.8165
0.4763	-0.8165
0.4763	-0.4082
0.0680	-0.4082
0.0680	0.4082
0.4763	0.4082
0.4763	0.8165
-0.3402	0.8164

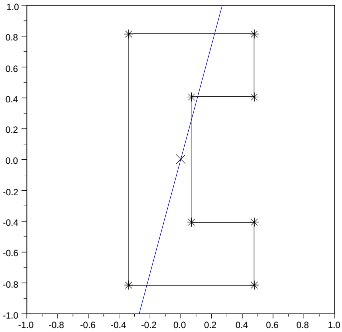
Non-convex polygon : C-shape



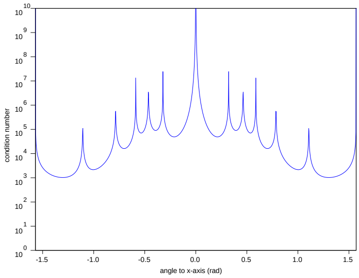
Estimation of the number of vertices



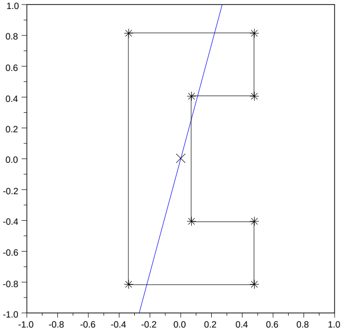
Non-convex polygon : C-shape



Choice of a reference direction



Non-convex polygon : C-shape

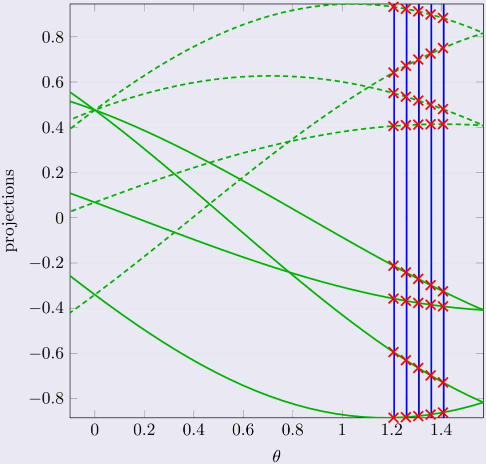


Pencil method for the reference direction

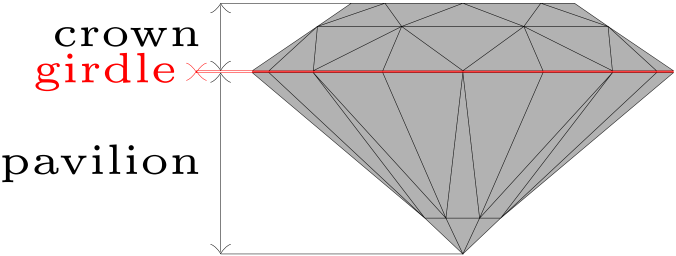
Reference direction	$(\cos\theta, \sin\theta)$
θ	1.30724
Conditioning	
$\kappa(W)$	$1.0\ 10^3$
Projections	$\mathcal{V}(\delta)$
0.9124	0.6997
0.5182	0.4119
-0.2701	-0.3764
-0.6642	-0.8769
Maximum error	
Err_w	$1.4\ 10^{-12}$

Non-convex polygon : C-shape

Interval interpolation

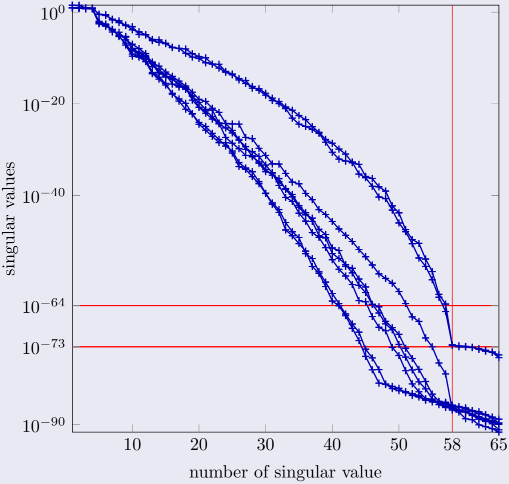


Diamond : polyhedron with 57 vertices



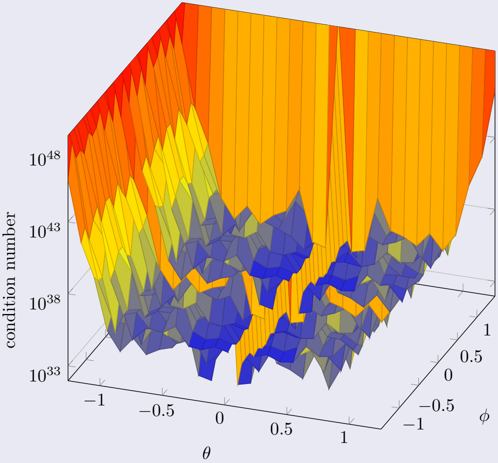
Diamond : polyhedron with 57 vertices

Estimation of the number of vertices using 70 digits

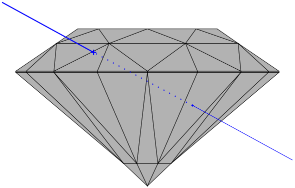


Diamond : polyhedron with 57 vertices

Choice of a reference direction



Diamond : polyhedron with 57 vertices

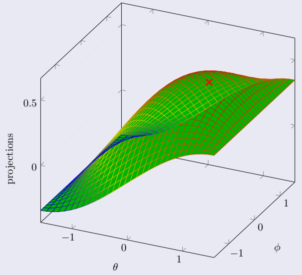


Pencil method for the reference direction

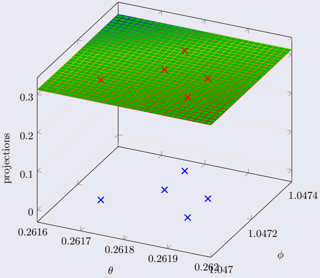
Reference direction	
$(\cos\theta\cos\phi,$	$\cos\theta\sin\phi, \sin\theta)$
(θ, ϕ)	$(0.2618, 1.0472)$
Conditioning	
$\kappa(W)$	$1.7\ 10^{33}$
Working precision	
ε	70
Maximum error	
Err_w	$1.9\ 10^{-9}$

Diamond : polyhedron with 57 vertices

Interval interpolation : one interpolant



Zoom in



Conclusion

At the moment, advantages

- less directions are needed than in previous polygon reconstruction (Milanfar *et al.* 95)
- generalization in any dimension with a robust matching process

Next directions

- finding a better optimization process for the reference direction
- solving similar problems, using multivariate methods (generalized Hankel matrices), like

  multivariate exponential interpolation,

  cubature formula.

References

- Gravin N., Lasserre J., Pasechnik D.V., Robins, S., *The inverse moment problem for convex polytopes*, Discrete Comput Geom (2012)
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Thanks for your attention !!