Cubic Column Relations in Truncated Moment Problems

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Abstract. Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data. A typical example of an inverse problem is the inversion of the Radon transform. Here a function (for example of two variables) is deduced from its integrals along all possible lines. This problem is intimately connected with image reconstruction for X-ray computerized tomography.

Moment problems are a special class of inverse problems. While the classical theory of moments dates back to the beginning of the 20th century, the systematic study of *truncated* moment problems began only a few years ago. In this talk we will first survey the elementary theory of truncated moment problems, and then focus on moment problems admitting cubic column relations.

For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a representing measure μ, it is necessary for the associated *moment matrix* $M(n)$ to be positive semidefinite, and for the corresponding *algebraic variety*, V_β, to satisfy rank $M(n) \leq$ card V_β as well as the following *consistency condition*: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on V_β, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In previous joint work with L. Fialkow and M. Möller, we proved that for the *extremal* case (rank $M(n) =$ card V_β), positivity and consistency are sufficient for the existence of a (unique, rank $M(n)$-atomic) representing measure.

In recent joint work with Seonguk Yoo we consider cubic column relations in $M(3)$ of the form (in complex notation) $Z^3 = itZ + u\bar{Z}$, where u and t are real numbers. For (u, t) in the interior of a real cone, we prove that the algebraic variety V_β consists of exactly 7 points, and we then apply the above mentioned solution of the extremal moment problem to obtain a necessary and sufficient condition for the existence of a representing measure. To check consistency, one needs a new representation theorem for sextic polynomials in Z and $\bar{Z}$ which vanish in the 7-point set V_β. Our proof of this representation theorem relies on two successive applications of the Fundamental Theorem of Linear Algebra. For other extremal moment matrices admitting cubic column relations, one can appeal to the Division Algorithm from real algebraic geometry to obtain similar representations.