Cubic Column Relations in Truncated Moment Problems

(joint work with Seonguk Yoo)

Raúl E. Curto

Quantum Computing Workshop on Inverse Moment Problem
National University of Singapore, December 11, 2013

A) Low-order polynomial approx. on subintervals of decreasing size

Commonly used Newton-Cotes formulas

T $n=1$ $\int_a^b f(x)\,dx = \frac{h}{2}[f(a)+f(b)] - \frac{h^3}{12} f''(\xi)$

S $n=2$ $\int_a^b f(x)\,dx = \frac{h}{3}[f(a)+4f(\frac{a+b}{2})+f(b)] - \frac{h^5}{90} f^{(4)}(\xi)$

$\frac{3}{8}$ $n=3$ $\int_a^b f(x)\,dx = \left\{ \begin{array}{c} \frac{3h}{8}[f(a)+3f(a+h)+3f(b-h)+f(b)] \\ -\frac{3h^5}{80} f^{(4)}(\xi) \end{array} \right.$

$n=4$ $\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \frac{2h}{45}[7f(a)+32f(a+h)+12f(\frac{a+b}{2}) \\ +32f(b-h)+7f(b)] - \frac{8h^7}{945} f^{(6)}(\xi) \end{array} \right.$

B) Polynomial approximation of increasing degree, using fewer, strategically-placed nodes

Definition

A quadrature (or cubature) rule of size p and precision m is a numerical integration formula which uses p nodes, is exact for all polynomials of degree at most m, and fails to recover the integral of some polynomial of degree $m+1$.

Gaussian Quadrature (size n, precision $2n-1$)

$\int_{-1}^{1} f(t)\,dt = \sum_{j=0}^{n-1} \rho_j f(t_j^{(n)})$ for every polynomial $f \in \mathbf{R}_{2n-1}[t]$

(Gaussian means minimum number of nodes possible)

Interpolating Equations:

$$\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 0 & k = 1, 3, ..., 2n-1 \\ \frac{2}{k+1} & k = 0, 2, ..., 2n-2 \end{cases}$$

Example: $n = 2$

$$\begin{cases} \rho_0 + \rho_1 & = & 2 \\ \rho_0 t_0 + \rho_1 t_1 & = & 0 \\ \rho_0 t_0^2 + \rho_1 t_1^2 & = & \frac{2}{3} \\ \rho_0 t_0^3 + \rho_1 t_1^3 & = & 0 \end{cases}$$

$\rho_0 = \rho_1 = 1;\ t_0 = -\frac{\sqrt{3}}{3},\ t_1 = \frac{\sqrt{3}}{3}.$

$$\int_{-1}^{1} \sum_{k=0}^{3} a_k t^k = \sum_{j=0}^{1} \rho_j \sum_{k=0}^{3} a_k t_j^k$$

NA textbooks prove this by using orthogonal Legendre polynomials

($t_0 < ... < t_{n-1}$ are the zeros of the nth Legendre polynomial)

(RC-L. Fialkow, 1990) Can do this as follows:

$\gamma_0 := 2,\ \gamma_1 := 0,\ \gamma_2 := \frac{2}{3},\ \gamma_3 := 0,\ \gamma_4 := \frac{2}{5}$, etc.

Assume n even, and form the Hankel matrix

$$H(n) := \begin{pmatrix} 2 & 0 & \frac{2}{3} & \cdots & 0 & \frac{2}{n+1} \\ 0 & \frac{2}{3} & 0 & \cdots & \frac{2}{n+1} & 0 \\ \frac{2}{3} & 0 & & \cdots & 0 & \frac{2}{n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{2}{n+1} & 0 & \cdots & \frac{2}{2n-1} & 0 \\ \frac{2}{n+1} & 0 & \frac{2}{n+3} & \cdots & 0 & \end{pmatrix},$$

label the columns $\mathit{1}, T, T^2, ...,$

observe that $T^n = \varphi_0 \mathit{1} + ... + \varphi_{n-1} T^{n-1}$,

build the polynomial

$$g(t) := t^n - (\varphi_0 + ... + \varphi_{n-1} t^{n-1}),$$

(non-iterative construction of Legendre polynomials) find its zeros ($t_0 < ... < t_{n-1}$),

and compute the densities using the Vandermonde system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ t_0^{n-1} & t_1^{n-1} & \cdots & t_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \cdots \\ \gamma_{n-1} \end{pmatrix}.$$

With K. Clouse (1996 honors undergrad at Iowa) we checked that Gaussian quadrature together with [Tot] makes approximation very precise:

$$\int_0^1 f(t)\,dt \cong \int_0^1 B_{2n-1}(f)(t)\,dt = \sum_{j=0}^{n-1} \rho_j B_{2n-1}(f)(t_j^{(n)})$$

for $f(t) = e^{-t^2},\ \sin(\sin x),\ \ln(x^2+1),\ \sin(e^x),\ \dfrac{e^x}{x},\ \sqrt{4+x^3},\ \dfrac{\sin x}{\pi + x}$.

The basic idea is to augment the original Hankel matrix by one row and one column at a time, preserving the rank (which a fortiori preserves positivity):

$$H(n) \prec H(n+1) \prec ...H(\infty)$$

Then define

$$\langle p, q \rangle_{H(\infty)} := (H(\infty)\widehat{p}, \widehat{q})_{\ell_2},$$

and show that

$$\langle p, q \rangle_{H(\infty)} = \int p\bar{q} \; d\mu$$

for some finitely atomic rep. meas., with supp $\mu = \mathcal{Z}(g)$.

Our operator-theoretic methods have allowed to solve the so-called truncated moment problem in the real line (Hausdorff, Stieltjes, Hamburger).

The Truncated Real Moment Problem

Given a family of real numbers β: $\beta_0, \beta_1, \ldots, \beta_{2n}$ with $\beta_0 > 0$, the **TMP** entails finding a positive Borel measure μ supported in the real line $\mathbb{R}$ such that

$$\beta_i = \int t^i \, d\mu \qquad (0 \leq i \leq 2n);$$

μ is called a **representing measure** for β.

Theorem

FULL MP (**Hamburger, 1920**)

$$\exists \mu \Leftrightarrow A(n) := (\beta_{i+j})_{i,j=0}^{n} \equiv \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0 \; \forall \quad n \geq 0.$$

Theorem

FULL MP (**Stieltjes, 1894**)

$$\exists \mu \textit{ with } \operatorname{supp} \mu \subseteq [0,+\infty)$$

$$\Leftrightarrow (\beta_{i+j})_{i,j=0}^{n} \geq 0 \textit{ and } (\beta_{i+j+1})_{i,j=0}^{n} \geq 0 \ \forall \ n \geq 0.$$

$$\begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ \beta_2 & \beta_3 & \beta_4 & \ddots & \cdots \\ \beta_3 & \beta_4 & \ddots & \ddots & \cdots \\ \beta_4 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \multicolumn{5}{c}{\text{(localizing matrix)}} \end{pmatrix} \geq 0$$

The Truncated Complex Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **rep. meas.** for γ.
- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0, \infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a, b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \ \ (1 \leq k \leq [\frac{n}{2}] + 1; \deg p_{k-1} \leq k-1).$$

- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

Applications

- Subnormal Operator Theory (unilateral weighted shifts)

  For $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$, the weighted shift W_α is subnormal if and only if the moment problem $\alpha_0^2\alpha_1^2\cdots\alpha_{k-1}^2 = \int s^k d\mu(s)$ is soluble.
- Physics (determination of contours)
- Computer Science (image recognition and reconstruction)
- Geography (location of proposed distribution centers)
- Probability (reconstruction of p.d.f.'s)

- Environmental Science (oil spills, via quadrature domains)

- Engineering (tomography)

- Optimization (finding the global minimum of a real polynomial in several real variables - J. Lasserre)

- Function Theory (a dilation-type structure theorem in Fejér-Riesz factorization theory - S. McCullough)

- Geophysics (inverse problems, cross sections)

Typical Problem: Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p in z and $\bar{z}$, $\deg p \leq n$

Given $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$0 \leq \int \mid p(z, \bar{z}) \mid^2 d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}.$$

- To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\textbf{“matricial” positivity}) \quad \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \dots & \dots & \dots & \dots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1)=\begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2)=\begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

In the real case, $\mathcal{M}(n)_{ij} := \gamma_{i+j},\ \ i,j \in \mathbb{Z}_+^2$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, ..., \gamma_{06}, ..., \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

A Version of Riesz-Haviland for TMP

Given a moment sequence β, the Riesz functional is

$$L_\beta(p) := p(\beta) \;\; (p \in \mathbb{C}[z, \bar{z}]).$$

Recall the Riesz-Haviland Theorem:

$\exists\ \mu$ rep. meas. for $\beta \Leftrightarrow L \equiv L_\beta \geq 0$ on $\mathcal{P}_+$.

- For TMP, the natural analogue won't work.
- We say that the Riesz functional L is *K-positive* if

$p \in \mathcal{P}$ and $p|K \geq 0 \Rightarrow L(p) \geq 0.$

- Consider the case

  $d = 1,\ K = \mathbb{R}$, and

$$M(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$$

  Then L_β is $\mathbb{R}$-positive, but no rep.meas. exists. For, in this case,

$$\mathrm{L}(\mathrm{a}_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4$$

  To see that L is $\mathbb{R}$-positive, recall that if $p \in \mathcal{P}_4$ satisfies $p|_{\mathbb{R}} \geq 0$, then there exist $f, g \in \mathcal{P}_2$ such that $p = f^2 + g^2$.

Now

$$L(p) = L(f^2 + g^2) = \langle M(2)\hat{f}, \hat{f}\rangle + \langle M(2)\hat{g}, \hat{g}\rangle \geq 0;$$

thus, L is $\mathbb{R}$-positive.

Assume that μ is a representing measure for β. Since

$$\int (x-1)^2 \, d\mu = L(x^2 - 2x + 1) = \beta_2 - 2\beta_1 + \beta_0 = 0,$$

it follows that $(x-1)|_{\text{supp } \mu} \equiv 0$. We thus have $(x-1)x^3|_{\text{supp } \mu} \equiv 0$, so

$$0 = \int (x-1)x^3 \, d\mu = L(x^4 - x^3) = \beta_4 - \beta_3 = 1,$$

a contradiction. Thus L is K-positive, but β has no representing measure.

In TMP, K-positivity is a necessary (but not sufficient) condition for a K-representing measure μ.

Theorem (TMP Version of Riesz-Haviland)

(RC-LF, 2007) $\beta \equiv \beta^{(2n)}$ *admits a K-representing measure if and only if L_β admits a K-positive linear extension* $L : \mathcal{P}_{2n+2} \longmapsto \mathbb{R}$.

This Theorem implies the classical Riesz-Haviland, via Stochel's Theorem.

$$
\begin{tikzcd}
\mathcal{P} \\
\vdots \\
\mathcal{P}_{2n+2} \arrow[u] \arrow[dr, "K\text{-pos.}"] \\
\mathcal{P}_{2n} \arrow[u] \arrow[r, "K\text{-pos.}"] & \mathbb{R}
\end{tikzcd}
$$

- **Main tool**. Let $\beta \equiv \beta^{(2n)}$ and let $K \subseteq \mathbb{R}^d$ be closed. Assume L_β is K-positive. Then $\beta \equiv \beta^{(2n-1)}$ has a K-representing measure.
- In general it is quite difficult to directly verify that an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive. One approach to establishing K-positivity or the existence of representing measures is through extensions of moment matrices.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases} .$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ =rank $A \Leftrightarrow C = W^*AW$.

Corollary

Assume $\operatorname{rank}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

We say that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is a flat extension of A. Observe that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}.$$

Corollary

Assume that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Then

$$\begin{aligned}
\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} &= \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - W^*AW \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}^* \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}^* \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}
\end{aligned}$$

(sum-of-squares representation).

For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij}\bar{z}^i z^j$, let $\widehat{p}$ denote the vector of coefficients and define

$$p(Z,\bar{Z}) := \sum a_{ij}\bar{Z}^i Z^j \equiv M(n)\widehat{p}.$$

If there exists a rep. meas. μ, then

$$p(Z,\bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) *If* $p, q, pq \in \mathcal{P}_n$, **and** $p(Z,\bar{Z}) = 0$,

then $(pq)(Z,\bar{Z}) = 0$.

Singular TMP; Real Case

- Given a finite family of moments, build moment matrix.
- Identify all column relations $p(Z, \bar{Z}) = 0$, i.e., $M(n)\widehat{p} = 0$.
- Build algebraic variety

$$\mathcal{V} := \bigcap_{p \in \mathcal{P}_n,\ \widehat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p .$$

- Always true: in the presence of a measure,

$$\text{supp } \mu \subseteq \mathcal{V}.$$

Therefore,

$$r := \operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \ \operatorname{supp} \ \mu \leq v := \operatorname{card} \mathcal{V}(\gamma).$$

Thus, if the variety is finite there's a natural candidate for supp μ, i.e., supp $\mu = \mathcal{V}(\gamma)$

(**It is possible for the inclusion** supp $\mu \subseteq \mathcal{V}$ **to be proper**.)

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Strategy: Build positive extension, repeat, and eventually extremal

  rank $M(n) \leq$ rank $M(n+1) \leq$ card $\mathcal{V}(M(n+1)) \leq$ card $\mathcal{V}(M(n))$
- General case.

First Existence Criterion

Theorem

(RC-L. Fialkow, 1998) Let γ be a truncated moment sequence. TFAE:

(i) γ *has a rep. meas.;*

(ii) γ *has a rep. meas. with moments of all orders;*

(iii) γ *has a compactly supported rep. meas.;*

(iv) γ *has a finitely atomic rep. meas. (with at most* $(n+2)(2n+3)$ *atoms);*

(v) $M(n) \geq 0$ *and for some* $k \geq 0$ $M(n)$ *admits a positive extension* $M(n+k)$*, which in turn admits a flat (i.e., rank-preserving) extension* $M(n+k+1)$ *(here* $k \leq 2n^2+6n+6$ *).*

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^* M(n-1) & W^* M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank $M(n)$*-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r :=$rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + ... + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + ... + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)
- $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies supp $\mu \subseteq \mathcal{Z}(p)$.

When $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \mathrm{Re}[z]$ and $y := \mathrm{Im}[z]$.

Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

$$
\begin{array}{lll}
(a) & \bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W & \text{parabola; } y = x^2 \\
(b) & \bar{W}^2 = -4i1 + W^2 & \text{hyperbola; } yx = 1 \\
(c) & \bar{W}^2 = W^2 & \text{pair of intersect. lines; } yx = 0 \\
(d) & \bar{W}W = 1 & \text{unit circle; } x^2 + y^2 = 1.
\end{array}
$$

Theorem QUARTIC

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ *be given, and assume* $M(2) \geq 0$ *and* $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. *Then* $\gamma^{(4)}$ *admits a rep. meas.* μ. *Moreover, it is possible to find* μ *with* card supp $\mu =$ rank $M(2)$, *except in some cases when* $\mathcal{V}(\gamma^{(4)})$ *is a pair of intersecting lines, in which cases there exist* μ *with* card supp $\mu \leq 6$.

Corollary

Assume that $M(2) \geq 0$, $M(2)$ *singular, and that* rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then* $M(2)$ *admits a representing measure.*

Extremal Real MP; $r = v$

The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \widehat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$.

- If β admits a representing measure μ, then

$$p \in \mathcal{P}_n \text{ satisfies } \widehat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

Easy Example:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Observe that $X = 1$, so $\mathcal{V} = \{1\}$, and therefore $r = 2$ and $v = 1$. It follows that this TMP admits no representing measure.

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \; d\mu = 0$.

Here Λ is the Riesz functional, given by $\Lambda(\bar{z}^i z^j) := \gamma_{ij}$

Basic necessary conditions for the existence of a representing measure

$$\text{(Positivity)}\ \mathcal{M}(n) \geq 0 \tag{8.1}$$

$$\text{(Consistency)}\ p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{8.2}$$

$$\text{(Variety Condition)}\ r \leq v, \text{ i.e., } \operatorname{rank}\ \mathcal{M}(n) \leq \operatorname{card}\ \mathcal{V}. \tag{8.3}$$

Consistency implies

$$\text{(Recursiveness)}\ p, q, pq \in \mathcal{P}_n,\ \widehat{p} \in \ker \mathcal{M}(n) \Longrightarrow \widehat{pq} \in \ker \mathcal{M}(n). \tag{8.4}$$

(ideal-like property)

Previous results:

- For $d = 1$ (the T *Hamburger* MP for $\mathbb{R}$), positivity and recursiveness are sufficient
- For $d = 2$, there exists $\mathcal{M}(3) > 0$ for which β has no representing measure
- In general, *Positivity*, *Consistency* and the *Variety Condition* are **not** sufficient.

Question C

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, β is consistent, and $r \leq v$, does β admit a representing measure?

The next result gives an affirmative answer to Question C in the *extremal* case, i.e., $r = v$.

Theorem EXT

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$,
the following are equivalent:

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $\mathcal{M}(n)$*-atomic (minimal);*

(iii) $\mathcal{M}(n) \geq 0$ *and* β *is consistent.*

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$.

Recall

Theorem A

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and $\deg\ p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires checking that the C block in the extension satisfies the Toeplitz condition, something highly nontrivial.

Here's a different approach:

We'd like to study the case of harmonic poly's: $q(z, \bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$.

Recall that rank $M(n) \leq$ card $\mathcal{Z}(q)$

so of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP admits a flat extension, or has no representing measure. In the case when $g(z) \equiv z$, we have

LEMMA

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

Bézout's Theorem predicts card $\mathcal{Z}(f(z) - \bar{z}) \leq 9$

- To get 7 points is not easy, as most complex cubic harmonic poly's tend to have 5 or fewer zeros. One way to maximize the number of zeros is to impose symmetry conditions on the zero set K. Also, the substitution $w = z + b/3$ (which produces an equivalent TMP) transforms a cubic $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; WLOG, we always assume that there's no quadratic term in the analytic piece.

- Now, for a poly of the form $z^3 + \alpha z + \beta \bar{z}$, it is clear that $0 \in K$ and that $z \in K \Rightarrow -z \in K$. Another natural condition is to require that K be symmetric with respect to the line $y = x$, which in complex notation is $z = i\bar{z}$. When this is required, we obtain $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$. Thus, the column relation becomes $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$.

Under these conditions, one needs to find only two points, one on the line $y = x$, the other outside that line.

We thus consider the harmonic polynomial $q_7(z,\bar{z}) := z^3 - itz - u\bar{z}$.

Proposition

(RC-S. Yoo, '09) card $\mathcal{Z}(q_7) = 7$. *In fact, for* $0 < |u| < t < 2\,|u|$,

$$\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\},$$

where $p, q, r > 0$, $p^2 + q^2 = u$ *and* $r^2 = \frac{t-u}{2}$.

To prove this result, we first identify the two real poly's Re $q_7 = x^3 - 3xy^2 + ty - ux$ and Im $q_7 = -y^3 + 3x^2y - tx + uy$ and calculate $\mathit{Resultant}(\mathit{Req}_7, \mathit{Imq}_7, y)$, which is the determinant of the Sylvester matrix, i.e.,

$$\begin{aligned} &\det \begin{pmatrix} -3x & t & x^3 - ux & 0 & 0 \\ 0 & -3x & t & x^3 - ux & 0 \\ 0 & 0 & -3x & t & x^3 - ux \\ -1 & 0 & 3x^2 + u & -tx & 0 \\ 0 & -1 & 0 & 3x^2 + u & -tx \end{pmatrix} \\ = \quad & x\left(u - t + 2x^2\right)\left(u + t + 2x^2\right)\left(16x^4 - 16x^2u + t^2\right). \end{aligned}$$

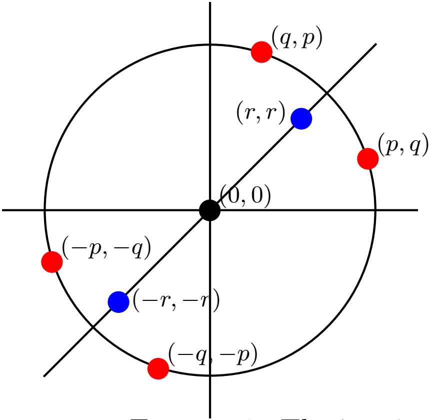

Figure 1. The 7-point set $\mathcal{Z}(q_7)$, where $r = \sqrt{\frac{t-u}{2}}$, $p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2})$ and $p^2 + q^2 = u$

The fact that q_7 has the maximum number of zeros predicted by the Lemma is significant to us, in that each sextic TMP with invertible $M(2)$ and a column relation of the form $q_7(Z, \bar{Z}) = 0$ either does not admit a representing measure or is necessarily extremal.

As a consequence, the existence of a representing measure will be established once we prove that such a TMP is consistent. This means that for each poly p of degree at most 6 that vanishes on $\mathcal{Z}(q_7)$ we must verify that $\Lambda(p) = 0$.

Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2 Z$. An analysis using the properties of the functional calculus shows that, **in the presence of a representing measure**, the new column relation must be

$$\bar{Z}^2 Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$

Notation

In what follows, $\mathbb{C}_6[z,\bar{z}]$ will denote the space of complex polynomials in z and $\bar{z}$ of degree at most 6, and let

$$\begin{aligned} q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= i(z - i\bar{z})(\bar{z}z - u). \end{aligned}$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

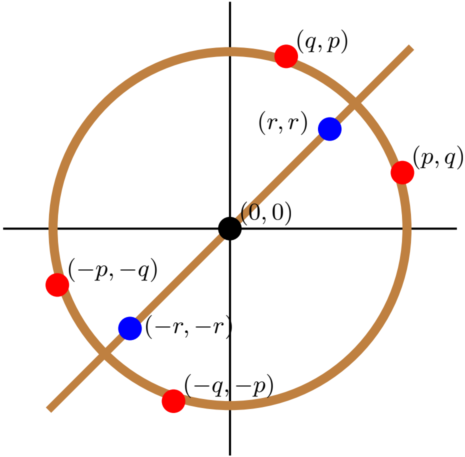

FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Main Theorem (RC & S. Yoo, J. Funct. Anal., 2014)

Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z, \bar{Z}) = 0$. There exists a representing measure for $M(3)$ if and only if

$$\left\{ \begin{array}{rcl} \Lambda(q_{LC}) & = & 0 \\ \Lambda(zq_{LC}) & = & 0. \end{array} \right. \tag{9.1}$$

Equivalently,

$$\left\{ \begin{array}{rcl} Re\ \gamma_{12} - Im\ \gamma_{12} = u(Re\ \gamma_{01} - Im\ \gamma_{01}) & = & 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\ Im\ \gamma_{02} & = & 0. \end{array} \right.$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0 \tag{9.2}$$

Proof. ($\Longrightarrow$) Let μ be a representing measure. We know that $7 \leq \text{rank } M(3) \leq \text{card supp } \mu \leq \text{card } \mathcal{Z}(q_7) = 7$, so that supp $\mu = \mathcal{Z}(q_7)$ and rank $M(3) = 7$. Thus,

$$\Lambda(q_7) = \int q_7 \, d\mu = 0.$$

Similarly, since supp $\mu \subseteq \mathcal{Z}(q_{LC})$, we also have

$$\Lambda(q_{LC}) = \Lambda(z q_{LC}) = 0,$$

as desired.

$(\Longleftarrow)$ On $\mathcal{Z}(q_7)$ we have $z^3 = itz + u\bar{z}$. Using this relation and (9.1), we can prove that $\Lambda(\bar{z}^i z^j q_{LC}) = 0$ for all $0 \leq i + j \leq 3$. For example,

$$\begin{aligned} \bar{z}q_{LC} - izq_{LC} &= (\bar{z} - iz)(\bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z}) \\ &= -uz^2 + \bar{z}z^3 - u\bar{z}^2 + \bar{z}^3 z \\ &= -uz^2 + \bar{z}(itz + u\bar{z}) - u\bar{z}^2 + (-it\bar{z} + uz)z \\ &= 0, \end{aligned}$$

and therefore $\Lambda(\bar{z}q_{LC}) = i\Lambda(zq_{LC}) = 0$. It follows that for $f, g, h \in \mathbb{C}_3[z, \bar{z}]$ we have $\Lambda(fq_7 + g\bar{q}_7 + hq_{LC}) = 0$. Consistency will be established once we show that all degree-six polynomials vanishing in $\mathcal{Z}(q_7)$ are of the form $fq_7 + g\bar{q}_7 + hq_{LC}$.

Let T be a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$, and consider the exact sequence

$$0 \to \text{ker } T \hookrightarrow \mathcal{X} \to \text{Ran } T \to 0.$$

Then

$$\text{dim ker } T - \text{dim } \mathcal{X} + \text{dim } RanT = 0.$$

Equivalently,

$$\text{dim } \mathcal{X} = \text{dim ker } T + \text{dim Ran } T.$$

Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ *and let*
$\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC}$ *for some* $f, g, h \in \mathbb{C}_3[z,\bar{z}]\}$.
Then $\mathcal{P}_6 = \mathcal{I}$.

Proof. Clearly, $\mathcal{I} \subseteq \mathcal{P}_6$. We shall show that $\dim \mathcal{I} = \dim \mathcal{P}_6$. Let $T : \mathbb{C}^{30} \longrightarrow \mathbb{C}_6[z,\bar{z}]$ be given by

$$(a_{00}, \cdots, a_{30}, b_{00}, \cdots, b_{30}, c_{00}, \cdots, c_{30}) \longmapsto$$

$$\begin{aligned} &(a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7 \\ +&(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7 \\ +&(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}. \end{aligned}$$

Recall that $30 = \dim\ \mathbb{C}^{30} = \dim\ \ker\ T + \dim \operatorname{Ran}\ T$, and observe that $\mathcal{I} = \operatorname{Ran}\ \ T$, so that $\dim\ \mathcal{I} = \operatorname{rank}\ \ T$.

To determine rank T, we first determine dim ker T. Using Gaussian elimination, we prove that $\dim\ \ker\ \ T = 9$ whenever $ut \neq 0$. It follows that rank $T = 30 - 9 = 21$, that is, $\dim\ \mathcal{I} = 21$.

Now consider the evaluation map $S : \mathbb{C}_6[z,\bar{z}] \longrightarrow \mathbb{C}^7$ given by

$$
\begin{aligned}
S(p(z,\bar{z})) \quad := \quad & (p(w_0,\bar{w}_0), p(w_1,\bar{w}_1), p(w_2,\bar{w}_2), \\
& p(w_3,\bar{w}_3), p(w_4,\bar{w}_4), p(w_5,\bar{w}_5), p(w_6,\bar{w}_6)).
\end{aligned}
$$

Again, $\dim \ker S + \dim \operatorname{Ran} S = \dim \mathbb{C}_6[z,\bar{z}] = 28$. Using Lagrange Interpolation, it is easy to verify that S is onto, i.e., $\operatorname{rank} S = 7$. Moreover, $\ker S = \mathcal{P}_6$. Since $\dim \mathbb{C}_6[z,\bar{z}] = 28$, it follows that $\dim \ker S = 21$, and a fortiori that $\dim \mathcal{P}_6 = 21$.

Therefore, $\dim \mathcal{I} = 21 = \dim \mathcal{P}_6$, and since $\mathcal{I} \subseteq \mathcal{P}_6$, we have established that $\mathcal{I} = \mathcal{P}_6$, as desired.

Yet Another Approach to TMP: The Division Algorithm

Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$

Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have

$$multideg(f) \geq multideg(a_i f_i).$$

- The Division Algorithm work is as follows: we identify sufficiently many polynomials $f_1, \cdots, f_s$ vanishing on $\mathcal{V}(\beta)$, and simultaneously in the kernel of the Riesz functional L_β. By the Division Algorithm, any polynomial f vanishing on $\mathcal{V}(\beta)$ can be written as $f = a_1 f_1 + \cdots + a_s f_s + r$, which readily implies that r must also vanish on $\mathcal{V}(\beta)$. Due to the divisibility condition on the monomials of r, and the characteristics of $\mathcal{V}(\beta)$, which generate an invertible Vandermonde matrix, we then prove that $r \equiv 0$.

- With some additional work, it is then possible to prove that $f \in \ker \ L_\beta$, which establishes the Consistency of β.

Summary

- Given a finite family of moments, build moment matrix
- Identify all column relations, and build algebraic variety $\mathcal{V}$
- Always true: $r \leq \text{card supp } \mu \leq v$
- Finite rank case; flat case
- Quartic Case
- Extremal case (must check Consistency)
- Harmonic cubic poly's in Sextic Case
- General singular case
- Invertible case still a big mystery...

The Case of Invertible $M(2)$

(L. Fialkow and J. Nie, 2010) Consider a quartic moment problem with invertible $M(2)$. Then there exists a representing measure.

The proof is abstract, using convex analysis.

(RC and S. Yoo, 2013) Concrete construction of a representing measure, when $M(2)$ is invertible. Moreover, there exists a 6-atomic representing measure, that is, $M(2)$ admits a flat extension $M(3)$.

Some References

- K. Atkinson, An Introduction to Numerical Analysis, John Wiley & Sons, 1992
- C. Bayer and J. Teichmann, The proof of Tchakaloff's Theorem, Proc. Amer. Math. Soc. 134(2006)
- R. Cools and P. Rabinowitz, Monomial cubature rules since "Stroud": a compilation, J. Comp. Appl. Math. 48(1993)
- RC, An operator-theoretic approach to truncated moment problems, Banach Center Publ. 38(1997)
- RC & L. Fialkow, Recursiveness, positivity, and truncated moment problems, Houston J. Math. 17(1991)

- RC & L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, IEOT 17(1993)
- RC & L. Fialkow, Solution of the truncated complex moment problem with flat data, Memoirs AMS 568(1996)
- RC & L. Fialkow, Flat extensions of positive moment matrices: Relations in analytic or conjugate terms, Operator Th.: Adv. Appl.104(1998)
- RC & L. Fialkow, Flat extensions of positive moment matrices: Recursively generated relations, Memoirs AMS 648(1998)
- RC & L. Fialkow, The truncated complex K-moment problem, Transactions Amer. Math. Soc. 352(2000)
- RC & L. Fialkow, The quadratic moment problem for the unit disk and unit circle, Integral Equations Operator Theory 38(2000)

- RC & L. Fialkow, Solution of the singular quartic moment problem, J. Operator Theory 48(2002)
- RC & L. Fialkow, A duality proof of Tchakaloff's Theorem, JMAA 269(2002)
- RC & L. Fialkow, Solution of the truncated parabolic moment problem, Integral Equations Operator Theory 50(2004)
- RC & L. Fialkow, Solution of the truncated hyperbolic moment problem, Integral Equations Operator Theory 52(2005)
- RC & M. Putinar, Nearly subnormal operators and moment problems, J. Funct. Anal. 115(1993)

- RC & S. Yoo, Cubic column relations in truncated moment problems, J. Funct. Anal. (2014)
- P.J. Davis and P. Rabinowitz, Methods of Numerical Integration, Academic Press, 1994
- L. Fialkow, Positivity, extensions and the truncated complex moment problem, Contemporary Math. 185(1995)
- L. Fialkow, Minimal representing measures arising from rank-increasing moment matrix extensions, JOT (2000)
- L. Fialkow and J. Nie, Positivity of Riesz functionals and solutions of quadratic and quartic moment problems, J. Funct. Anal. (2010)
- M. Putinar, Extremal solutions of the two-dimensional L-problem of moments, J. Funct. Anal. 136(1996)

- M. Putinar, A note on Tchakaloff's Theorem, PAMS 125(1997)
- M. Putinar, Linear analysis of quadrature domains, Ark. för Mat. 33(1995)
- M. Putinar, A dilation theory approach to cubature formulas, Expo. Math. 15(1997)
- M. Putinar and F.-H. Vasilescu, Problème des moments sur les compacts semi-algébriques, C.R. Acad. Sci. Paris 323(1996)
- M. Putinar and F.-H. Vasilescu, Solving moment problems by dimensional extension, Annals of Math. 149(1999)
- K. Schmüdgen, An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional, Math. Nachr. 88(1979)

- K. Schmüdgen, The K-moment problem for compact semi-algebraic sets, Math. Ann. 289(1991)
- J.L. Smul'jan, An operator Hellinger integral (Russian), Mat. Sb. 91(1959)
- J. Stochel, Solving the truncated moment problem solves the moment problem, Glasgow J. Math. 43(2001)
- J. Stochel and F. Szafraniec, Algebraic operators and moments on algebraic sets, Portugaliae Math 51(1994)
- A.H. Stroud, Approximate Calculation of Multiple Integrals, 1971
- V. Tchakaloff, Formules de cubatures mécaniques à coefficients non négatifs, Bull. Sc. Math. 81(1957)

- V. Totik, Approximation by Bernstein polynomials, Amer. J. Math. 116(1994)
- F.-H. Vasilescu, Moment problems for multi-sequences of operators, J. Math. Anal. Appl. 219(1998)
- Wolfram Research, Inc., *Mathematica*, Version 2.1, Champaign, IL, 1992