# Minimum norm points on polytope boundaries

David Bremner

UNB

December 20, 2013

Joint work with Yan Cui, Zhan Gao

# Outline

# Support functionals

## Linear Programs with varying objectives

$$\langle x, y \rangle = x^T y = y^T x$$

For $K \subset \mathbb{R}^d, w \in \mathbb{R}^d$,

$$\sup(K; w) := \sup_{k \in K} \langle w, k \rangle \quad \text{(upper)}$$

$$\inf(K; w) := \inf_{k \in K} \langle w, k \rangle \quad \text{(lower)}$$

$$= -\sup(K; -w)$$

Let polyhedron $K = \{ Ax \leq b \}$

$$\sup(K; c) = \max_{Ax \leq b} c^T x$$

# Support functionals

## Linear Programs with varying objectives

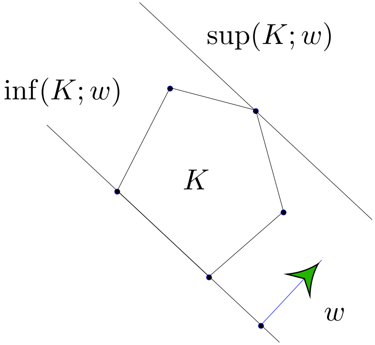
$$\langle x, y \rangle = x^T y = y^T x$$

For $K \subset \mathbb{R}^d, w \in \mathbb{R}^d$,

$$\sup(K; w) := \sup_{k \in K} \langle w, k \rangle \quad \text{(upper)}$$

$$\inf(K; w) := \inf_{k \in K} \langle w, k \rangle \quad \text{(lower)}$$

$$= -\sup(K; -w)$$




Let polyhedron $K = \{ Ax \leq b \}$

$$\sup(K; c) = \max_{Ax \leq b} c^T x$$

# Support functionals

## Linear Programs with varying objectives

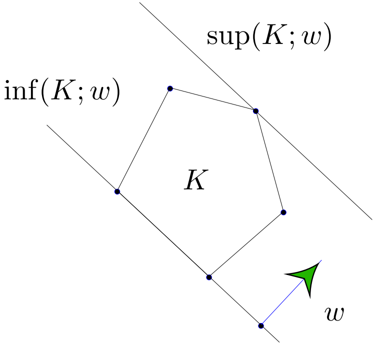
$$\langle x, y \rangle = x^T y = y^T x$$

For $K \subset \mathbb{R}^d, w \in \mathbb{R}^d$,

$$\sup(K; w) := \sup_{k \in K} \langle w, k \rangle \quad \text{(upper)}$$

$$\inf(K; w) := \inf_{k \in K} \langle w, k \rangle \quad \text{(lower)}$$

$$= -\sup(K; -w)$$




Let polyhedron $K = \{ Ax \le b \}$

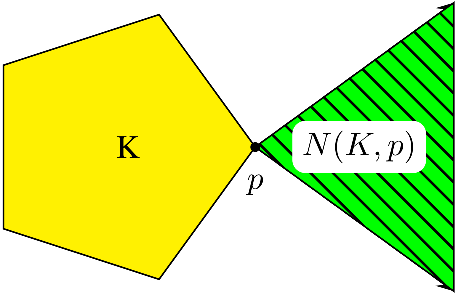
$$\sup(K; c) = \max_{Ax \le b} c^T x$$

# Outer Normal Cone

## Geometric LP Duality

For $K \subset \mathbb{R}^d, p \in K$, the *outer normal cone*

$$N(K;p) := \{\, y \mid \langle\, y, p\,\rangle = \sup(K;y)\,\}$$



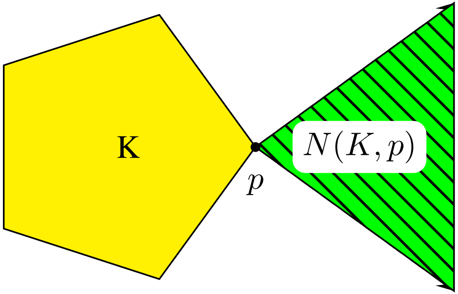

- $N(K;p)$ is the set of all objective functions optimal at $x$

# Outer Normal Cone

## Geometric LP Duality

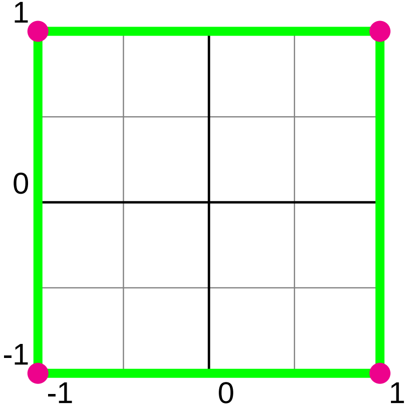
For $K \subset \mathbb{R}^d, p \in K$, the *outer normal cone*

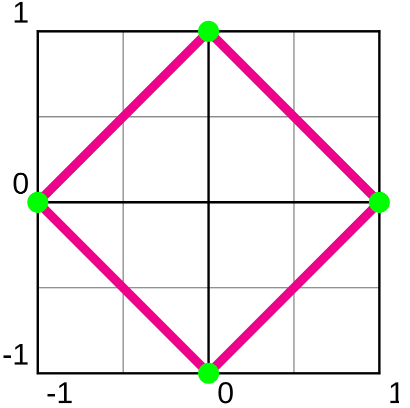
$$N(K;p) := \{\, y \mid \langle\, y, p \,\rangle = \sup(K;y) \,\}$$




- $N(K;p)$ is the set of all objective functions optimal at $x$

# Polarity




$$P^* = \text{conv}\,\{\, y \mid \forall x \in P, \langle\, y, x \,\rangle \leq 1 \,\}$$

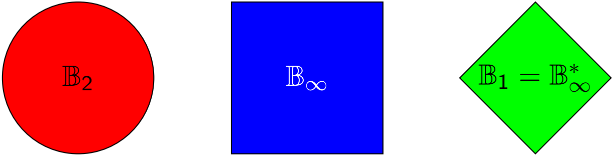
# Minkowski Norms

$$\|x\|_{\mathbb{B}} = \inf \{ \gamma \geq 0 \mid x \in \gamma\mathbb{B} \} \tag{1}$$
$$= \sup(\mathbb{B}^*; x) \tag{2}$$

# Minkowski Norms

$$\|x\|_{\mathbb{B}} = \inf \{ \gamma \geq 0 \mid x \in \gamma\mathbb{B} \} \tag{1}$$

$$= \sup(\mathbb{B}^*; x) \tag{2}$$



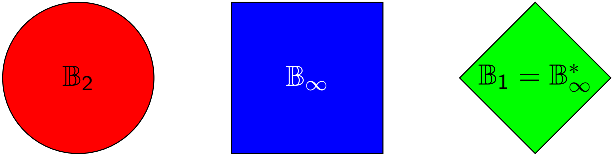

$\| \cdot \|_2 = \sqrt{x_1^2 + x_2^2}$

$\| \cdot \|_\infty = \max(|x_1|, |x_2|)$

$\| \cdot \|_1 = |x_1| + |x_2|$

# Minkowski Norms

$$\|x\|_{\mathbb{B}} = \inf \{ \gamma \geq 0 \mid x \in \gamma\mathbb{B} \} \quad (1)$$
$$= \sup(\mathbb{B}^*; x) \quad (2)$$



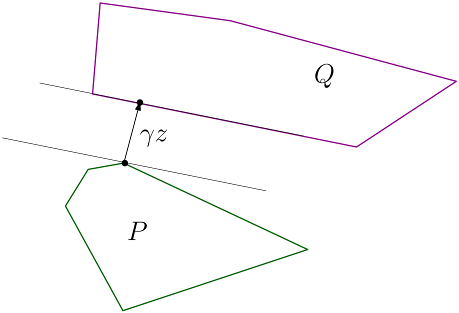

$\|\cdot\|_2 = \sqrt{x_1^2 + x_2^2}$ $\quad$ $\|\cdot\|_\infty = \max(|x_1|, |x_2|)$ $\quad$ $\|\cdot\|_1 = |x_1| + |x_2|$

$\mathbb{S} = \operatorname{bd}\mathbb{B}$ $\quad$ $\mathbb{S}^* = \operatorname{bd}(\mathbb{B}^*)$

# Margin

## Separable

$P, Q \subset \mathbb{R}^d$ are *separable* if $\exists z \neq \vec{0}$ such that $\sup(Q; z) \leq \inf(P; z)$.




## Margin

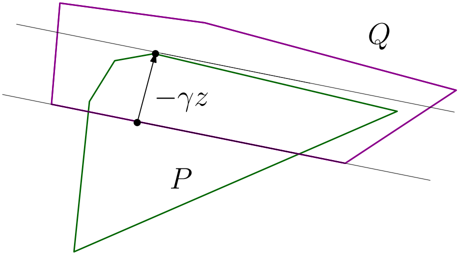
$$\max \gamma \quad \text{such that}$$
$$\exists z \in \mathbb{S}^*$$
$$\sup(Q; z) + \gamma \leq \inf(P; z)$$

- For the Euclidean norm $z \in \mathbb{S}^*$ just means $z$ is a unit vector.
- Both that constraint and $z \neq \vec{0}$ are non-convex

# Margin

## Separable

$P, Q \subset \mathbb{R}^d$ are *separable* if $\exists z \neq \vec{0}$ such that $\sup(Q; z) \leq \inf(P; z)$.




## Margin

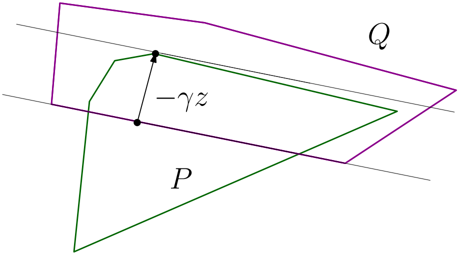
$$\max \gamma \quad \text{such that}$$
$$\exists z \in \mathbb{S}^*$$
$$\sup(Q; z) + \gamma \leq \inf(P; z)$$

- For the Euclidean norm $z \in \mathbb{S}^*$ just means $z$ is a unit vector.
- Both that constraint and $z \neq \vec{0}$ are non-convex

# Margin

## Separable

$P, Q \subset \mathbb{R}^d$ are *separable* if $\exists z \neq \vec{0}$ such that $\sup(Q; z) \leq \inf(P; z)$.




## Margin

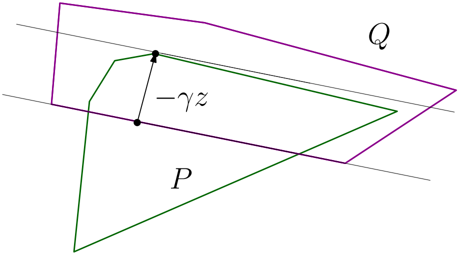
$$\max \gamma \quad \text{such that}$$
$$\exists z \in \mathbb{S}^*$$
$$\sup(Q; z) + \gamma \leq \inf(P; z)$$

- For the Euclidean norm $z \in \mathbb{S}^*$ just means $z$ is a unit vector.
- Both that constraint and $z \neq \vec{0}$ are non-convex

# Margin

## Separable

$P, Q \subset \mathbb{R}^d$ are *separable* if $\exists z \neq \vec{0}$ such that $\sup(Q; z) \leq \inf(P; z)$.




## Margin

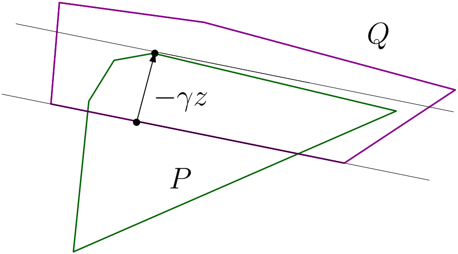
$$\max \gamma \quad \text{such that}$$
$$\exists z \in \mathbb{S}^*$$
$$\sup(Q; z) + \gamma \leq \inf(P; z)$$

- For the Euclidean norm $z \in \mathbb{S}^*$ just means $z$ is a unit vector.
- Both that constraint and $z \neq \vec{0}$ are non-convex

# Margin

## Separable

$P, Q \subset \mathbb{R}^d$ are *separable* if $\exists z \neq \vec{0}$ such that $\sup(Q; z) \leq \inf(P; z)$.




## Margin

$$\max \gamma \quad \text{such that}$$
$$\exists z \in \mathbb{S}^*$$
$$\sup(Q; z) + \gamma \leq \inf(P; z)$$

- For the Euclidean norm $z \in \mathbb{S}^*$ just means $z$ is a unit vector.
- Both that constraint and $z \neq \vec{0}$ are non-convex

# A simplified convex approximation

$$\min \|w\|^2 + C\gamma$$
$$s.t.$$
$$\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q$$
$$\gamma \geq 0$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$w' = \rho w$$
$$\gamma' = 1 - \rho(1 - \gamma)$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

# A simplified convex approximation

$$\min \|w\|^2 + C\gamma$$
$$s.t.$$
$$\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q$$
$$\gamma \geq 0$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$w' = \rho w$$
$$\gamma' = 1 - \rho(1 - \gamma)$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

# A simplified convex approximation

$$\min \|w\|^2 + C\gamma$$
$$s.t.$$
$$\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q$$
$$\gamma \geq 0$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$w' = \rho w$$
$$\gamma' = 1 - \rho(1 - \gamma)$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

# A simplified convex approximation

$$\min \|w\|^2 + C\gamma$$
$$s.t.$$
$$\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q$$
$$\gamma \geq 0$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$w' = \rho w$$
$$\gamma' = 1 - \rho(1 - \gamma)$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

# A simplified convex approximation

$$\min \|w\|^2 + C\gamma$$
$$s.t.$$
$$\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q$$
$$\gamma \geq 0$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$w' = \rho w$$
$$\gamma' = 1 - \rho(1 - \gamma)$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

# A simplified convex approximation

$$
\begin{gathered}
\min \|w\|^2 + C\gamma \\
s.t. \\
\langle w, p - q \rangle \geq 1 - \gamma \quad p \in P, q \in Q \\
\gamma \geq 0
\end{gathered}
$$

- Need $C$ suff. large to force $\gamma < 1$ at opt.
- Given optimal $(w, \gamma < 1)$, $\rho > 1$, define

$$
\begin{aligned}
w' &= \rho w \\
\gamma' &= 1 - \rho(1 - \gamma)
\end{aligned}
$$

- If $\gamma < 1$

$$\gamma - \gamma' = (\rho - 1)(1 - \gamma) > 0$$

- Therefore

$$\|w'\| - C\gamma' \leq \|w\| + C\gamma$$

- For large enough $C$, we can scale $w$ up without making the objective worse.

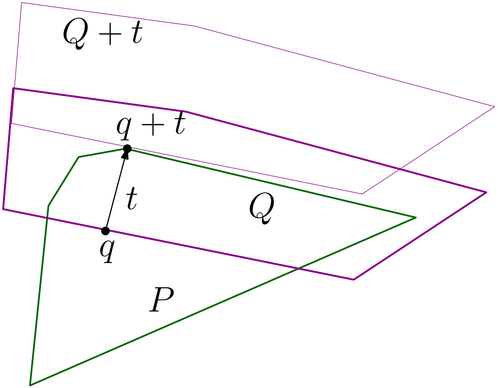
# Translate until separable

## Minimize Translation

Let $P$ and $Q$ convex bodies.

$$\begin{aligned} \inf \quad & \|t\| \\ \sup(P; z) &\leq \inf(Q + t; z) \\ z \neq \vec{0}, \quad q &\in Q, \quad q + t \in P \end{aligned}$$

$$\begin{aligned} \inf \quad & \|r\| \\ z &\in N(P - Q; r) \\ z \neq \vec{0}, \quad r &\in P - Q \end{aligned}$$




## Equivalent Formulation

$$\operatorname{shift}(P, Q) := \min \{ \|t\| : t \in \operatorname{bd}(P - Q) \}$$

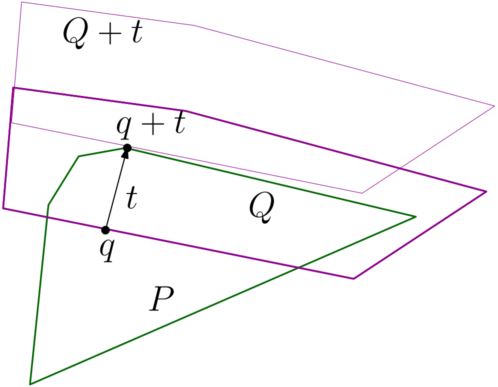
# Translate until separable

## Minimize Translation

Let $P$ and $Q$ convex bodies.

$$\inf \quad \|t\|$$
$$\sup(P; z) \leq \inf(Q + t; z)$$
$$z \neq \vec{0}, \quad q \in Q, \quad q + t \in P$$

$$\inf \quad \|r\|$$
$$z \in N(P - Q; r)$$
$$z \neq \vec{0}, \quad r \in P - Q$$




## Equivalent Formulation

$$\text{shift}(P, Q) := \min \{ \|t\| : t \in \text{bd}(P - Q) \}$$

# Duality

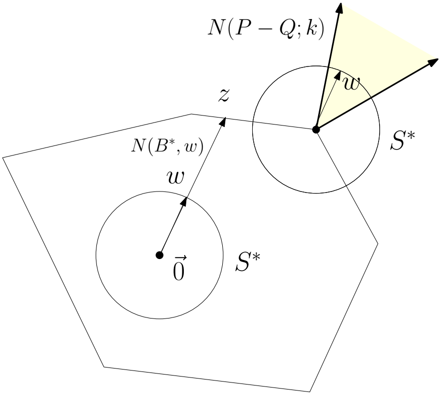
**Proposition**

*If $\vec{0} \in P - Q$,*

$$\text{shift}(P, Q) = -\text{margin}(P, Q)$$

Choose $w \in \mathbb{S}^*$,
$z \in N(\mathbb{B}^*; w) \cap \text{bd}(P - Q)$

$$\begin{aligned}\sup(P - Q, w) &\geq \langle z, w \rangle = \|z\| \\ &\geq \inf_{k \in \text{bd}\, P - Q} \|k\|\end{aligned}$$




If $k \in \text{bd}\, P - Q$,
$w \in N(P - Q; k) \cap \mathbb{S}^*$

$$\|k\| \geq \langle w, k \rangle = \sup(P - Q; w)$$

# Duality

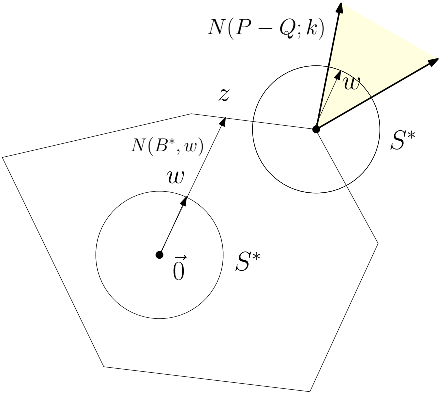
## Proposition

*If* $\vec{0} \in P - Q$,

$$\text{shift}(P, Q) = -\,\text{margin}(P, Q)$$

Choose $w \in \mathbb{S}^*$,
$z \in N(\mathbb{B}^*; w) \cap \text{bd}(P - Q)$

$$\sup(P - Q, w) \geq \langle\, z, w \,\rangle = \|z\| \geq \inf_{k \in \text{bd}\, P - Q} \|k\|$$




If $k \in \text{bd}\, P - Q$,
$w \in N(P - Q; k) \cap \mathbb{S}^*$

$$\|k\| \geq \langle\, w, k \,\rangle = \sup(P - Q; w)$$

# Duality

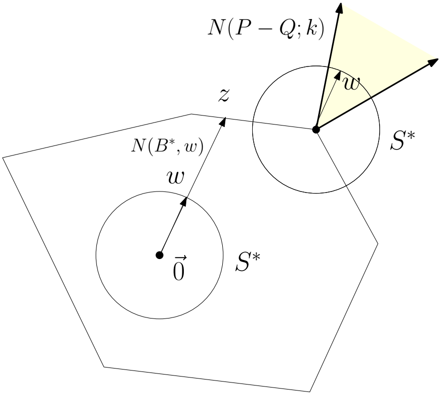
**Proposition**

*If* $\vec{0} \in P - Q$,

$$\operatorname{shift}(P, Q) = -\operatorname{margin}(P, Q)$$

Choose $w \in \mathbb{S}^*$,
$z \in N(\mathbb{B}^*; w) \cap \operatorname{bd}(P - Q)$

$$\begin{aligned}\sup(P - Q, w) &\geq \langle z, w \rangle = \|z\| \\ &\geq \inf_{k \in \operatorname{bd} P - Q} \|k\|\end{aligned}$$



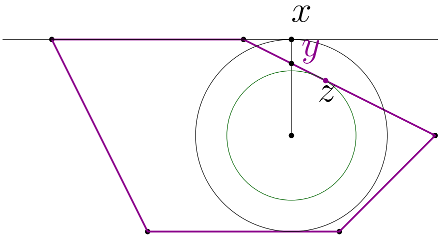

If $k \in \operatorname{bd} P - Q$,
$w \in N(P - Q; k) \cap \mathbb{S}^*$

$$\|k\| \geq \langle w, k \rangle = \sup(P - Q; w)$$

# Convex Maximization

For convex polyhedron $K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \min_{F \text{ facet of } K} \min_{r \in F} \|r\|$$
$$= \min_{F \text{ facet of } K} \min_{r \in \text{aff}\, F} \|r\|$$




$$\min \{ x \mid a^T x = 1 \} = \frac{1}{\|a\|}$$

Assuming $\vec{0} \in \text{int}\, K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \max_{v \text{ vertex of } K^*} \|v\|$$
$$= \max_{x \in K^*} \|x\|$$

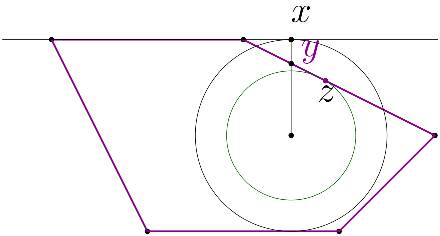
**Theorem (Brieden 2002)**

*For $2 \leq p < \infty$ $\max \|x\|_p^p \mid Ax \leq b$ cannot be computed in polynomial time within a factor of 1.090 , unless P=NP.*

# Convex Maximization

For convex polyhedron $K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \min_{F \text{ facet of } K} \min_{r \in F} \|r\|$$
$$= \min_{F \text{ facet of } K} \min_{r \in \text{aff}\, F} \|r\|$$




$$\min \{ x \mid a^T x = 1 \} = \frac{1}{\|a\|}$$

Assuming $\vec{0} \in \text{int}\, K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \max_{v \text{ vertex of } K^*} \|v\|$$
$$= \max_{x \in K^*} \|x\|$$

**Theorem (Brieden 2002)**

*For $2 \leq p < \infty$ $\max \|x\|_p^p \mid Ax \leq b$ cannot be computed in polynomial time within a factor of $1.090$, unless P=NP.*

# Convex Maximization

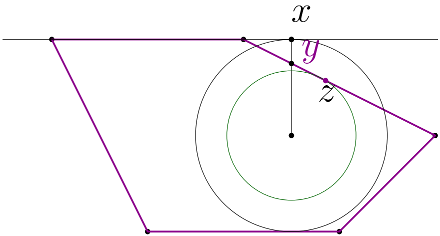
For convex polyhedron $K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \min_{F \text{ facet of } K} \min_{r \in F} \|r\|$$
$$= \min_{F \text{ facet of } K} \min_{r \in \text{aff}\, F} \|r\|$$

$$\min \{\, x \mid a^T x = 1 \,\} = \frac{1}{\|a\|}$$

Assuming $\vec{0} \in \text{int}\, K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \max_{v \text{ vertex of } K^*} \|v\|$$
$$= \max_{x \in K^*} \|x\|$$




**Theorem (Brieden 2002)**

*For $2 \leq p < \infty$ $\max \|x\|_p^p \mid Ax \leq b$ cannot be computed in polynomial time within a factor of $1.090$, unless P=NP.*

# Convex Maximization

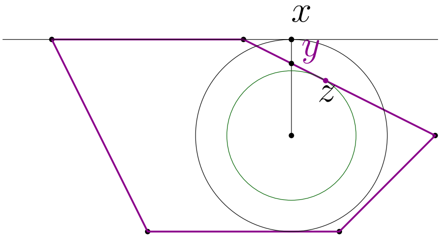
For convex polyhedron $K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \min_{F \text{ facet of } K} \min_{r \in F} \|r\|$$
$$= \min_{F \text{ facet of } K} \min_{r \in \text{aff}\, F} \|r\|$$

$$\min \{ x \mid a^T x = 1 \} = \frac{1}{\|a\|}$$

Assuming $\vec{0} \in \text{int}\, K$

$$\inf_{r \in \text{bd}\, K} \|r\| = \max_{v \text{ vertex of } K^*} \|v\|$$
$$= \max_{x \in K^*} \|x\|$$




**Theorem (Brieden 2002)**

*For $2 \leq p < \infty$ $\max \|x\|_p^p \mid Ax \leq b$ cannot be computed in polynomial time within a factor of 1.090 , unless P=NP.*

# Hardness

$$\text{breadth}_w(X) := \sup_{x,y \in X} \langle\, w, x - y \,\rangle = \sup(X - X; w)$$

$$\text{width}(X) := \inf_{w \in \mathbb{S}^*} \text{breadth}_w(X)$$

## Proposition

$$\text{width}(X) = -\,\text{margin}(X, X)$$

## Proposition (Gritzmann–Klee 1992)

*Computing the width is NP-hard.*

# Hardness

$$\mathrm{breadth}_w(X) := \sup_{x,y \in X} \langle\, w, x - y \,\rangle = \sup(X - X; w)$$

$$\mathrm{width}(X) := \inf_{w \in \mathbb{S}^*} \mathrm{breadth}_w(X)$$

## Proposition

$$\mathrm{width}(X) = -\,\mathrm{margin}(X, X)$$

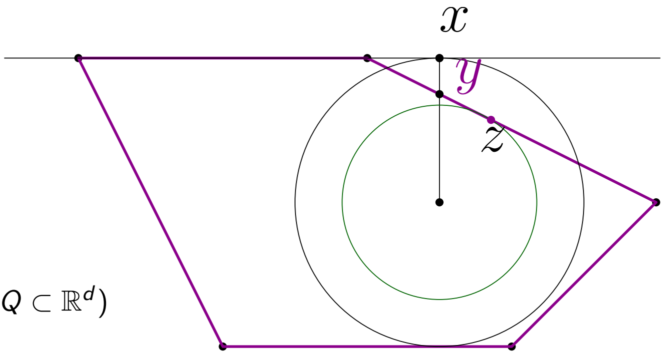
## Proposition (Gritzmann–Klee 1992)

*Computing the width is NP-hard.*

# Brute Force Solution

**function**
$\textsc{MinNormFacet}(P, Q \subset \mathbb{R}^d)$
  $\lambda = +\infty$
  $R = \mathrm{ExtremePoints}(P - Q)$
  **for all** facets $F$ of conv $R$ **do**
    $\lambda = \min(\mathrm{MinNormAff}(F), \lambda)$
  **end for**
**end function**

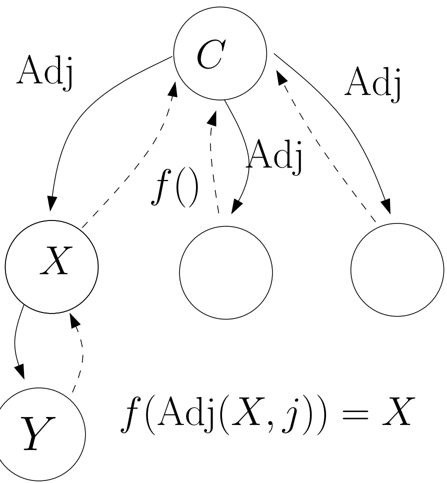
# Brute Force Solution




**function** $\textsc{MinNormFacet}(P, Q \subset \mathbb{R}^d)$
  $\lambda = +\infty$
  $R = \mathrm{ExtremePoints}(P - Q)$
  **for all** facets $F$ of conv $R$ **do**
    $\lambda = \min(\mathrm{MinNormAff}(F), \lambda)$
  **end for**
**end function**

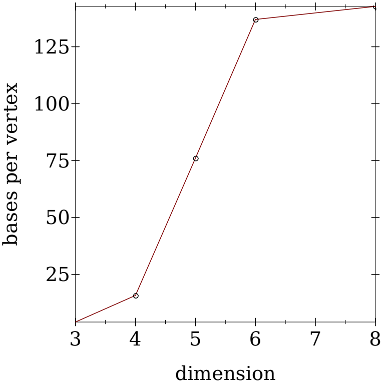
# Enumerating Facets with Reverse Search

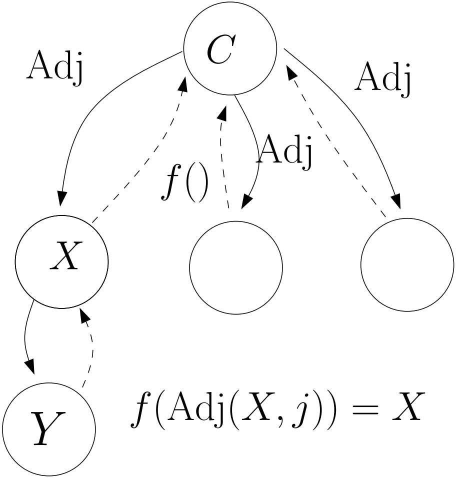
- Facets of $P - Q$ are computed without extra memory
- the number of facets and bases grow quickly.



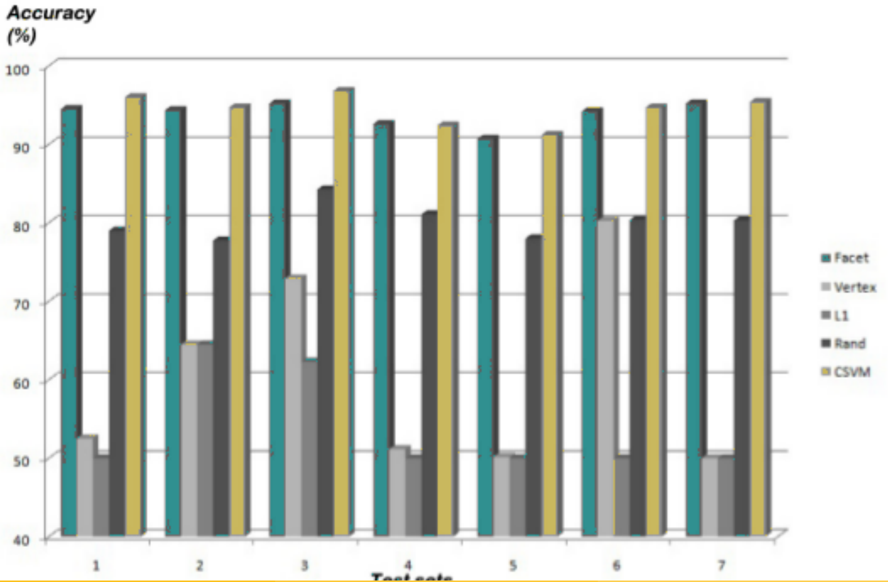
# Enumerating Facets with Reverse Search

- Facets of $P - Q$ are computed without extra memory
- the number of facets and bases grow quickly.

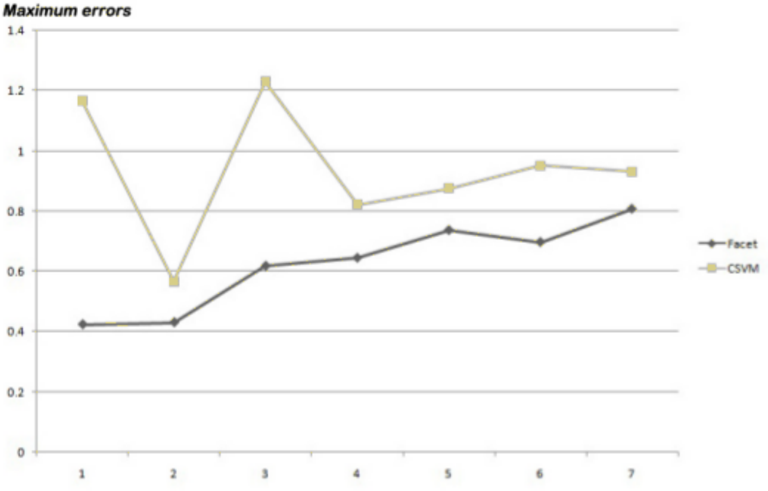


# Classification Accuracy

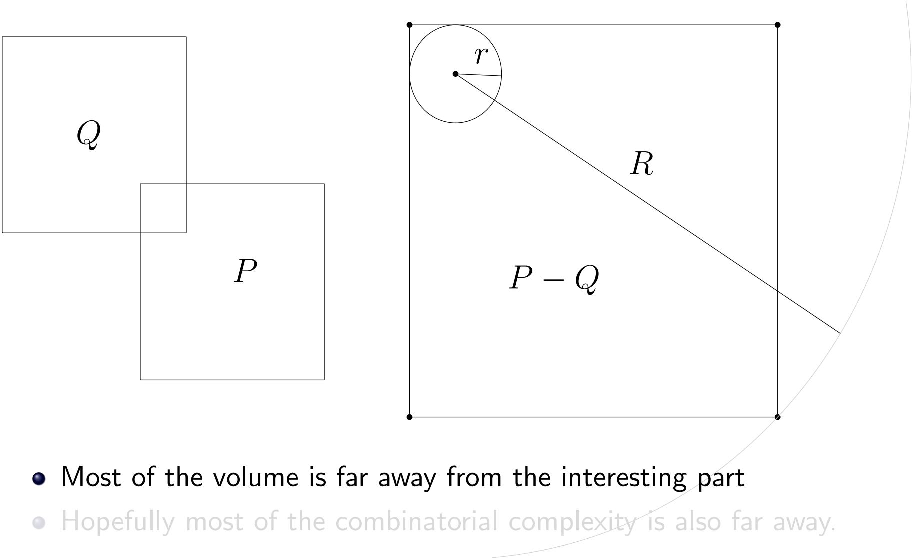


# Maximum Error



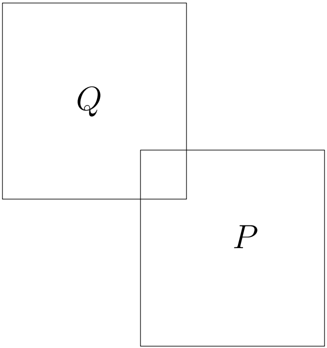

*Test Set*

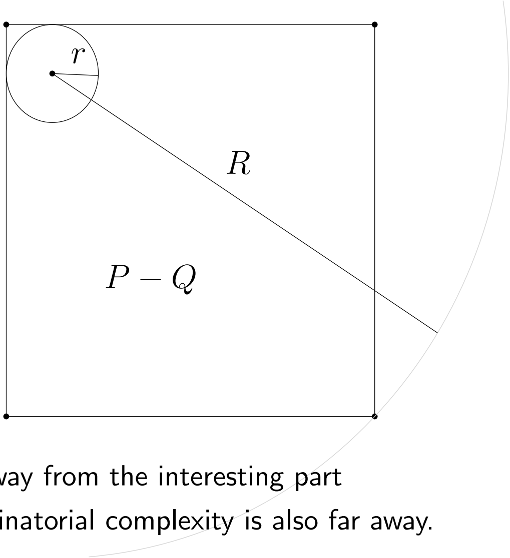
# Eccentricity




- Most of the volume is far away from the interesting part
- Hopefully most of the combinatorial complexity is also far away.

# Eccentricity



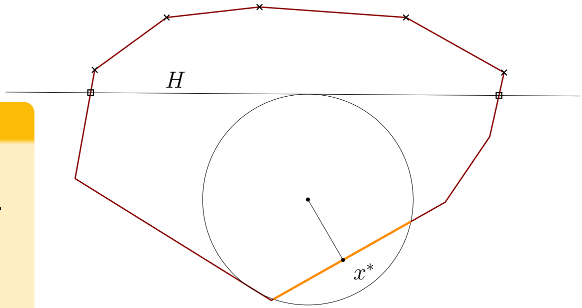




- Most of the volume is far away from the interesting part
- Hopefully most of the combinatorial complexity is also far away.

# Cutting Planes

## Cut

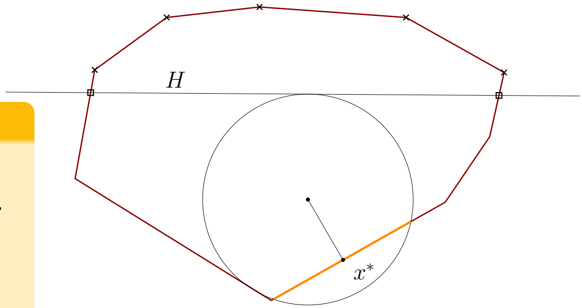
Let
$x^* = \text{argmin}_{x \in \text{bd}\, P-Q} \|x\|$.
For any $\lambda \geq \|x^*\|$, $H$ tangent to $\lambda\mathbb{B}$ preserves $\text{bd}(P-Q) \cap \lambda B \ni x^*$,




## Double description style update (Motzkin)

Let $E_H$ be the edges of conv $V$ that properly intersect $H$. The vertex set of $\text{conv}(V) \cap H^-$ is $V \cap H^- \cup \bigcup_{e \in E_H}(e \cap H)$

# Cutting Planes




## Cut

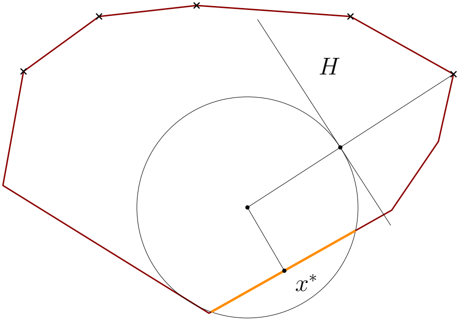
Let
$x^* = \text{argmin}_{x \in \text{bd}\, P-Q} \|x\|$.
For any $\lambda \geq \|x^*\|$, $H$
tangent to $\lambda\mathbb{B}$ preserves
$\text{bd}(P-Q) \cap \lambda B \ni x^*$,

## Double description style update (Motzkin)

Let $E_H$ be the edges of conv $V$ that properly intersect $H$. The vertex set of $\text{conv}(V) \cap H^-$ is $V \cap H^- \cup \bigcup_{e \in E_H} (e \cap H)$

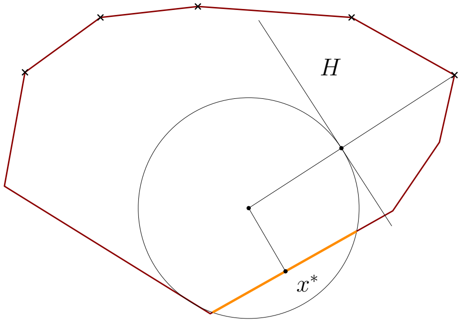
# Heuristics

- Adding a cutting plane is (usually) only effective if it cuts off many more vertices than it creates.
- Experimentally, the farthest vertex seems to be a good choice of cutting plane normal, at least for "random-ish" data.



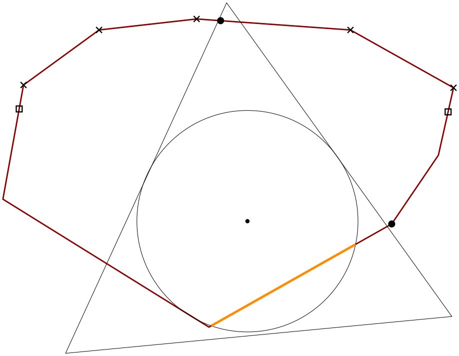
# Heuristics

- Adding a cutting plane is (usually) only effective if it cuts off many more vertices than it creates.
- Experimentally, the farthest vertex seems to be a good choice of cutting plane normal, at least for “random-ish” data.



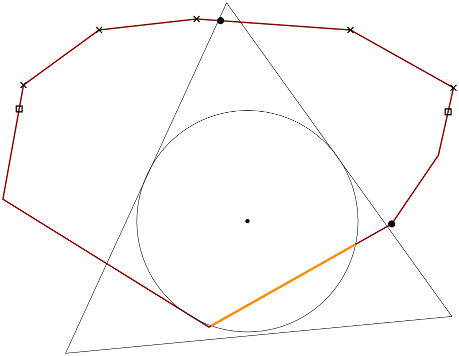
# Cutting Polytopes

- In some sense what we want is polytopal approximation of the sphere.
- Find polytope with small $H$ and $V$ complexity containing sphere.
- Using (approximately) regular simplex
- round carefully
- doesn't seem to iterate that well.



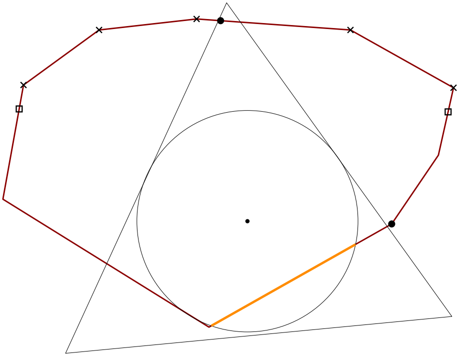
# Cutting Polytopes

- In some sense what we want is polytopal approximation of the sphere.
- Find polytope with small $H$ and $V$ complexity containing sphere.
- Using (approximately) regular simplex
- round carefully
- doesn't seem to iterate that well.



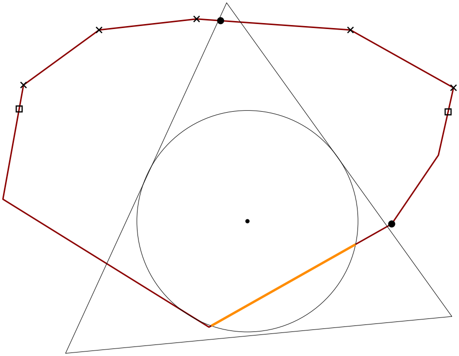
# Cutting Polytopes

- In some sense what we want is polytopal approximation of the sphere.
- Find polytope with small $H$ and $V$ complexity containing sphere.
- Using (approximately) regular simplex
- round carefully
- doesn't seem to iterate that well.



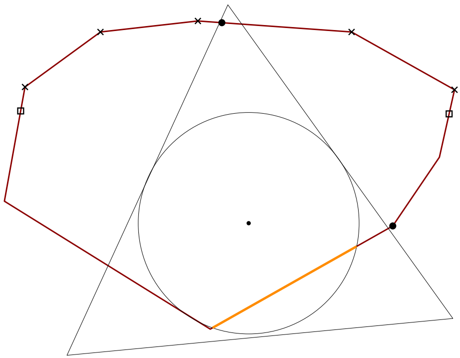
# Cutting Polytopes

- In some sense what we want is polytopal approximation of the sphere.
- Find polytope with small $H$ and $V$ complexity containing sphere.
- Using (approximately) regular simplex
- round carefully
- doesn't seem to iterate that well.

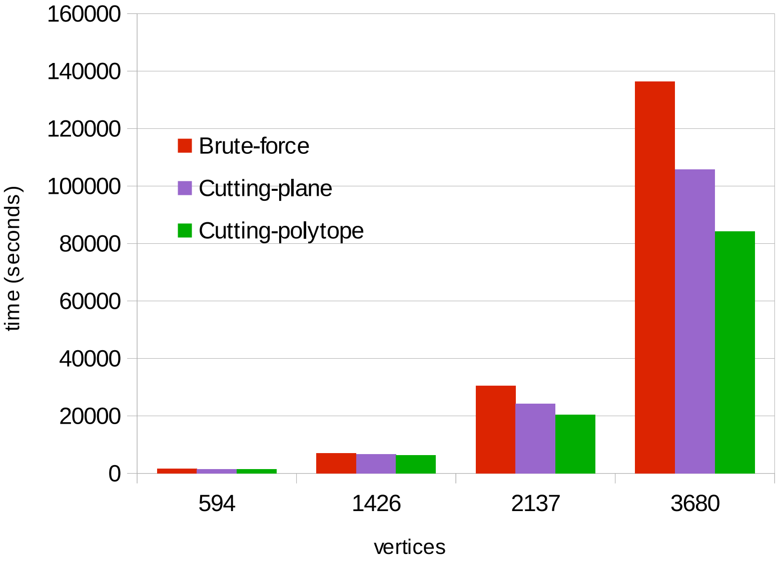


# Cutting Polytopes

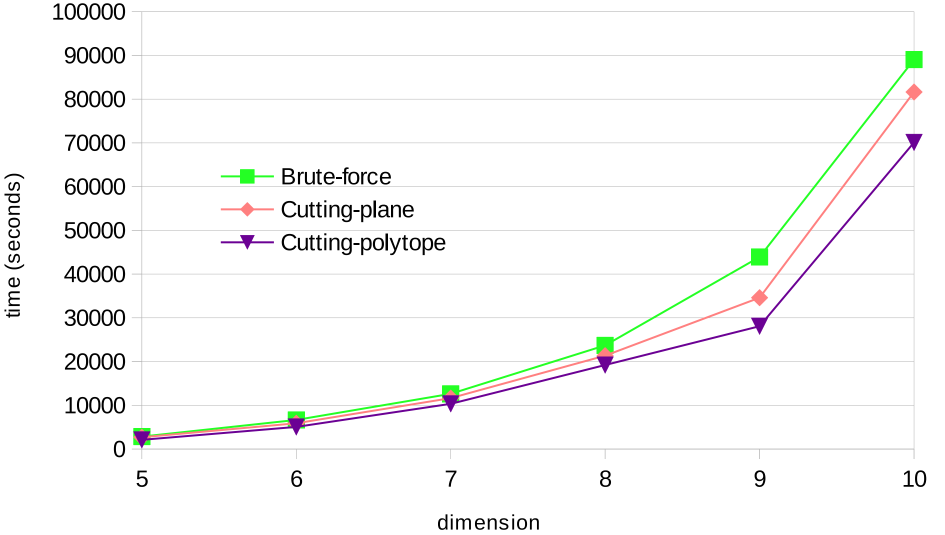
- In some sense what we want is polytopal approximation of the sphere.
- Find polytope with small $H$ and $V$ complexity containing sphere.
- Using (approximately) regular simplex
- round carefully
- doesn't seem to iterate that well.



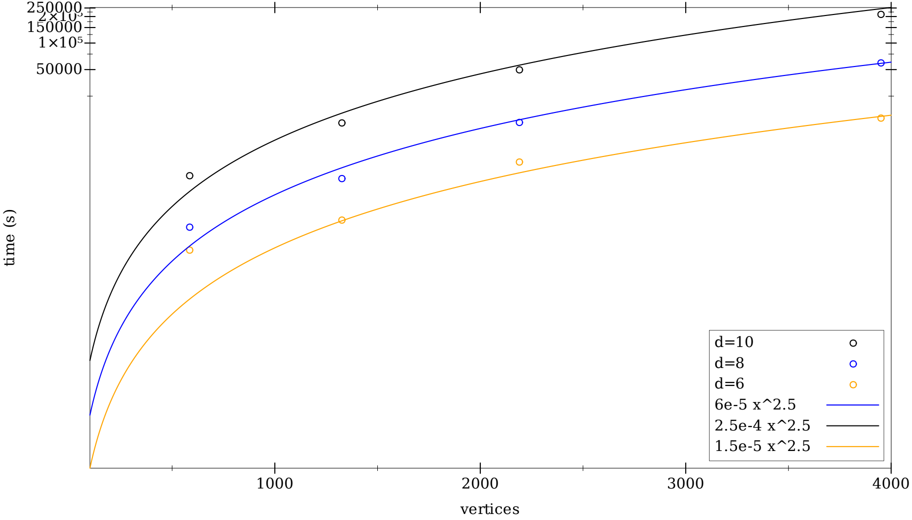
# Performance in $\mathbb{R}^9$



# Average runtime versus dimension



# Cutting polytope method



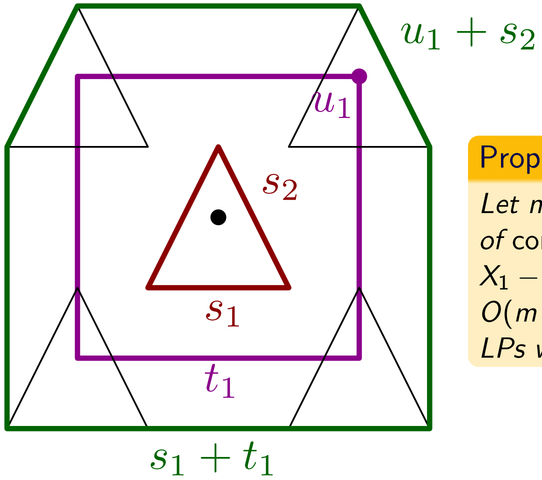
# Conclusions

- Minimum norm boundary point is dual to maximum margin classifying hyperplane
- NP-Hard
- Solvable via (partial) enumerative approaches.
- Cutting planes/polytopes yield some speedup, but plenty of room for improvement.

# Acknowledgements


- Geometric background: with Peter Gritzmann
- Software: lrslib (Avis), cddlib (Fukuda), gmp, Minkowski sum (Weibel)

# Filtering redundant points of $P - Q$




## Proposition (Fukuda)

*Let $m$ be the max degree of a vertex of $\mathrm{conv}(X_i)$. The extreme points of $X_1 - X_2$ can be computed with $O(m \cdot f_0(X_1 - X_2) + f_0^2(X_1) + f_0^2(X_2))$ LPs with $m$ constraints.*

# Vertices of the unit ball

**function** $\textsc{MinNormCut}(P, Q, \mathbb{B} \subset \mathbb{R}^d)$
    $\lambda = +\infty$
    **for all** vertices $v$ of $\mathbb{B}$ **do**
        $F \leftarrow \mathrm{SepFacet}(P - Q, \lambda v)$
        **if** $F \neq \emptyset$ **then**
            $\lambda = \min(\mathrm{MinNormAff}(F, \lambda))$
        **end if**
    **end for**
**end function**