Inverse trigonometric moment problem for piecewise-smooth functions

IMS Program on Inverse Moment Problems
Singapore, January 2014

Dmitry Batenkov

Weizmann Institute of Science
Rehovot, Israel

January 8th, 2014

Fourier inversion problem

$$f : S^1 \to \mathbb{R}$$

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\imath kx} f(x) dx$$

$$(f)_N = \sum_{|k| \leqslant N} c_k(f) e^{\imath kx}$$

$$(\Delta f)_N = f - (f)_N$$

- If $f \in C^d(S^1)$ then $\left|(\Delta f)_N\right| \sim N^{-d-1}$, uniformly.

Fourier inversion problem

$$f : S^1 \to \mathbb{R}$$
$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\imath kx} f(x)dx$$
$$(f)_N = \sum_{|k|\leqslant N} c_k(f)e^{\imath kx}$$
$$(\Delta f)_N = f - (f)_N$$

- If $f \in C^d(S^1)$ then $\left|(\Delta f)_N\right| \sim N^{-d-1}$, uniformly.
- If $f \in C^d(S^1 \setminus \{\xi\}_{j=1}^{p})$ then we have the *Gibbs phenomenon*:

Fourier inversion problem

$$f : S^1 \to \mathbb{R}$$
$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\imath kx} f(x) dx$$
$$(f)_N = \sum_{|k| \leqslant N} c_k(f) e^{\imath kx}$$
$$(\Delta f)_N = f - (f)_N$$

- If $f \in C^d(S^1)$ then $\left|(\Delta f)_N\right| \sim N^{-d-1}$, uniformly.
- If $f \in C^d(S^1 \setminus \{\xi\}_{j=1}^p)$ then we have the *Gibbs phenomenon*:
    - No uniform convergence

Fourier inversion problem

$$f : S^1 \to \mathbb{R}$$

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\imath kx} f(x)dx$$

$$(f)_N = \sum_{|k|\leqslant N} c_k(f) e^{\imath kx}$$

$$(\Delta f)_N = f - (f)_N$$

- If $f \in C^d(S^1)$ then $\left|(\Delta f)_N\right| \sim N^{-d-1}$, uniformly.
- If $f \in C^d(S^1 \setminus \{\xi\}_{j=1}^{p})$ then we have the *Gibbs phenomenon*:
    - No uniform convergence
    - $\left|(\Delta f)_N\right| \sim N^{-1}$ away from the jumps

Fourier inversion problem

$$f : S^1 \to \mathbb{R}$$

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\imath kx} f(x)dx$$

$$(f)_N = \sum_{|k|\leqslant N} c_k(f)e^{\imath kx}$$

$$(\Delta f)_N = f - (f)_N$$

- If $f \in C^d(S^1)$ then $\left|(\Delta f)_N\right| \sim N^{-d-1}$, uniformly.
- If $f \in C^d(S^1 \setminus \{\xi\}_{j=1}^{p})$ then we have the *Gibbs phenomenon*:
  - No uniform convergence
  - $\left|(\Delta f)_N\right| \sim N^{-1}$ away from the jumps
  - Implications: PDE, signal processing, imaging, ...

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

- Main problem: accurate determination of the jump locations $\{\xi\}_{j=1}^{p}$.

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

- Main problem: accurate determination of the jump locations $\{\xi\}_{j=1}^{p}$.
- Best possible accuracy: $O(N^{-d-2})$ for jumps, $O(N^{-d-1})$ for pointwise values.

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

- Main problem: accurate determination of the jump locations $\{\xi\}_{j=1}^{p}$.
- Best possible accuracy: $O(N^{-d-2})$ for jumps, $O(N^{-d-1})$ for pointwise values.
- Linear methods: no better than $O(N^{-1})$

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

- Main problem: accurate determination of the jump locations $\{\xi\}_{j=1}^{p}$.
- Best possible accuracy: $O(N^{-d-2})$ for jumps, $O(N^{-d-1})$ for pointwise values.
- Linear methods: no better than $O(N^{-1})$
- Tadmor et. al (concentration kernels): $O(N^{-1})$ for jumps, full accuracy between the jumps

Main problem

Question

Can piecewise-smooth functions be reconstructed from their Fourier coefficients with high accuracy?

- Main problem: accurate determination of the jump locations $\{\xi\}_{j=1}^{p}$.
- Best possible accuracy: $O(N^{-d-2})$ for jumps, $O(N^{-d-1})$ for pointwise values.
- Linear methods: no better than $O(N^{-1})$
- Tadmor et. al (concentration kernels): $O(N^{-1})$ for jumps, full accuracy between the jumps
- Our result: full accuracy for jumps and values in between.

Krylov-Gottlieb-Eckhoff method

$$f = \underbrace{\Phi}_{\text{piecewise polynomial}} + \underbrace{\Psi}_{\in C^d(S^1)}$$

$$c_k(f) = \underbrace{c_k(\Phi)}_{\text{finite-parametric}} + O\left(k^{-d-2}\right), \qquad k \gg 1.$$

- One gets a nonlinear system of algebraic equations of *Prony type*, with errors in the left-hand side.

Krylov-Gottlieb-Eckhoff method

$$
\begin{aligned}
f &= \underbrace{\Phi}_{\text{piecewise polynomial}} + \underbrace{\Psi}_{\in C^d(S^1)} \\
c_k(f) &= \underbrace{c_k(\Phi)}_{\text{finite-parametric}} + O\left(k^{-d-2}\right), \qquad k \gg 1.
\end{aligned}
$$

- One gets a nonlinear system of algebraic equations of *Prony type*, with errors in the left-hand side.
- Its stability analysis turns out to be hard.

Krylov-Gottlieb-Eckhoff method

$$f = \underbrace{\Phi}_{\text{piecewise polynomial}} + \underbrace{\Psi}_{\in C^d(S^1)}$$

$$c_k(f) = \underbrace{c_k(\Phi)}_{\text{finite-parametric}} + O\left(k^{-d-2}\right), \qquad k \gg 1.$$

- One gets a nonlinear system of algebraic equations of *Prony type*, with errors in the left-hand side.
- Its stability analysis turns out to be hard.

Eckhoff's conjecture (1995)

By solving the above system, one can reconstruct the jumps (positions + magnitudes), as well as the point-wise values of f, with the maximal (asymptotic) accuracy.

Eckhoff system

Start with piecewise polynomial Φ of degree d

$$D^{d+1}\Phi = \sum_{j=1}^{p} \sum_{\ell=0}^{d} c_{\ell,j} \delta^{(\ell)}(x - \xi_j)$$

$$(\imath k)^{d+1} c_k(\Phi) = c_k(D^{d+1}\Phi)$$

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad c_{\ell,j} \in \mathbb{R}, \xi_j \in S^1;$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2}.$$

Prony type systems

$$m_k = \sum_{j=1}^{p} z_j^k a_j$$

$$m_k = \sum_{j=1}^{p} \sum_{\ell=0}^{d_j-1} a_{\ell,j} z_j^{k-\ell} k(k-1)\cdots(k-\ell+1)$$

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}$$

- Frequency estimation, Padé approximation, array processing, statistics, interpolation, quadrature, radar signal detection, error correction codes, analytic continuation ...

Prony type systems

$$m_k = \sum_{j=1}^{p} z_j^k a_j$$

$$m_k = \sum_{j=1}^{p} \sum_{\ell=0}^{d_j-1} a_{\ell,j} z_j^{k-\ell} k(k-1)\cdots(k-\ell+1)$$

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}$$

- Frequency estimation, Padé approximation, array processing, statistics, interpolation, quadrature, radar signal detection, error correction codes, analytic continuation ...
- Sub-Nyquist Sampling (reconstruction of "spike trains")

Prony type systems

$$m_k = \sum_{j=1}^{p} z_j^k a_j$$

$$m_k = \sum_{j=1}^{p} \sum_{\ell=0}^{d_j-1} a_{\ell,j} z_j^{k-\ell} k(k-1)\cdots(k-\ell+1)$$

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}$$

- Frequency estimation, Padé approximation, array processing, statistics, interpolation, quadrature, radar signal detection, error correction codes, analytic continuation ...
- Sub-Nyquist Sampling (reconstruction of "spike trains")
- Shape reconstruction from moments (quadrature domains, ...)

Algebraic reconstruction

Method

Given: integral measurements of an unknown function f

$$m_k = \int f d\sigma_k \tag{1}$$

Algebraic reconstruction

Method

Given: integral measurements of an unknown function f

$$m_k = \int f d\sigma_k \tag{1}$$

1. Represent f by a finite-parametric model $F(p_1, \ldots, p_n)$.

Algebraic reconstruction

Method

Given: integral measurements of an unknown function f

$$m_k = \int f d\sigma_k \tag{1}$$

1. Represent f by a finite-parametric model $F(p_1, \dots, p_n)$.
2. Replace f with F in (1) and obtain system of (algebraic) equations

$$m_k = \int F d\sigma_k = G_k(p_1, \dots, p_n). \tag{2}$$

Algebraic reconstruction

Method

Given: integral measurements of an unknown function f

$$m_k = \int f d\sigma_k \tag{1}$$

1. Represent f by a finite-parametric model $F(p_1, \ldots, p_n)$.
2. Replace f with F in (1) and obtain system of (algebraic) equations
$$m_k = \int F d\sigma_k = G_k(p_1, \ldots, p_n). \tag{2}$$
3. Solve (2) in a robust way.

Algebraic reconstruction

Method

Given: integral measurements of an unknown function f

$$m_k = \int f d\sigma_k \tag{1}$$

1. Represent f by a finite-parametric model $F(p_1, \ldots, p_n)$.
2. Replace f with F in (1) and obtain system of (algebraic) equations
$$m_k = \int F d\sigma_k = G_k(p_1, \ldots, p_n). \tag{2}$$
3. Solve (2) in a robust way.

- Systems (2) will be of *Prony type*.

Solution methods

- Prony-like methods

Solution methods

- Prony-like methods
- Subspace-based/SVD (MUSIC, ESPRIT, Matrix pencils)

Solution methods

- Prony-like methods
- Subspace-based/SVD (MUSIC, ESPRIT, Matrix pencils)
- Least-squares methods

Solution methods

- Prony-like methods
- Subspace-based/SVD (MUSIC, ESPRIT, Matrix pencils)
- Least-squares methods
- Algebraic methods

Solution methods

- Prony-like methods
- Subspace-based/SVD (MUSIC, ESPRIT, Matrix pencils)
- Least-squares methods
- Algebraic methods
- Total variation minimization

Prony stability - open questions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell$$

Robust Prony solution

How robustly* can the parameters $\{a_{\ell,j}, z_j\}$ be recovered from the noisy data $\{\tilde{m}_k = m_k + \delta_k\}_{k=0}^{N-1}$?

* How does the error depend on $|\delta_k|, N$ and other data?

Prony stability - open questions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell$$

Robust Prony solution

How robustly* can the parameters $\{a_{\ell,j}, z_j\}$ be recovered from the noisy data $\{\tilde{m}_k = m_k + \delta_k\}_{k=0}^{N-1}$?

Superresolution

How robustly* can two closely spaced nodes $\{z_i, z_j\}$ be recovered from the noisy data $\{\tilde{m}_k = m_k + \delta_k\}_{k=0}^{N-1}$?

* How does the error depend on $|\delta_k|, N$ and other data?

Known lower bounds

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \qquad k = 0, 1, \ldots N$$

$$\Delta m_k \sim \varepsilon, \ \sum_j d_j = R$$

- Node separation: $|z_i - z_j| > \delta$

Known lower bounds

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \qquad k = 0, 1, \ldots N$$

$$\Delta m_k \sim \varepsilon, \ \sum_j d_j = R$$

- Node separation: $|z_i - z_j| > \delta$
- Donoho [1992]: for $d_j = 1$

$$\text{Error} \approx \left(\frac{1}{\delta}\right)^{2R+1} \varepsilon$$

Known lower bounds

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \qquad k = 0, 1, \ldots N$$

$$\Delta m_k \sim \varepsilon, \; \sum_j d_j = R$$

- Node separation: $|z_i - z_j| > \delta$
- Donoho [1992]: for $d_j = 1$

$$\text{Error} \approx \left(\frac{1}{\delta}\right)^{2R+1} \varepsilon$$

- Statistical estimation literature: as $N \gg 1$

$$\text{Error}\{z_j\} \approx \frac{1}{|a_{d_j-1,j}| N^{d_j}} \varepsilon$$

Prony systems: assumptions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k \in S \subseteq \{0, \ldots, N-1\}.$$

- Small perturbations: $\tilde{m}_k = m_k + \delta_k$ with $|\delta_k| \ll 1$.

Prony systems: assumptions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad k \in S \subseteq \{0, \ldots, N-1\}.$$

- Small perturbations: $\tilde{m}_k = m_k + \delta_k$ with $|\delta_k| \ll 1$.
- Number of equations $|S|$ = number of unknowns C.

Prony systems: assumptions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad k \in S \subseteq \{0, \ldots, N-1\}.$$

- Small perturbations: $\tilde{m}_k = m_k + \delta_k$ with $|\delta_k| \ll 1$.
- Number of equations $|S|$ = number of unknowns C.
- Stability measure is given by the Lipschitz constant of the "data $\to$ result mapping".

Prony systems: assumptions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad k \in S \subseteq \{0, \ldots, N-1\}.$$

- Small perturbations: $\tilde{m}_k = m_k + \delta_k$ with $|\delta_k| \ll 1$.
- Number of equations $|S|$ = number of unknowns C.
- Stability measure is given by the Lipschitz constant of the "data $\rightarrow$ result mapping".
  - Require row-wise norm estimates of the inverse Jacobian matrix.

Prony systems: assumptions

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k \in S \subseteq \{0, \ldots, N-1\}.$$

- Small perturbations: $\tilde{m}_k = m_k + \delta_k$ with $|\delta_k| \ll 1$.
- Number of equations $|S|$ = number of unknowns C.
- Stability measure is given by the Lipschitz constant of the "data $\to$ result mapping".
  - Require row-wise norm estimates of the inverse Jacobian matrix.
- Index subset S is an arithmetic progression with step σ.

Non-decimated Prony stability

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k = 0, 1, \ldots, C-1$$

$$\delta = \min_{i<j} |z_i - z_j|, \quad |\Delta m_k| < \varepsilon$$

$$\implies \qquad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta^{-C} \varepsilon$$

- Problem is ill-posed as $|a_{d_j-1,j}| \to 0$ and/or $\delta \to 0$.

D.Batenkov, Y.Yomdin, "On the accuracy of solving confluent Prony systems", *SIAM J.Appl.Math., 73(1), pp.134–154, 2013.*

Non-decimated Prony stability

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k = 0, 1, \ldots, C-1$$

$$\delta = \min_{i<j} |z_i - z_j|, \quad |\Delta m_k| < \varepsilon$$

$$\implies \qquad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta^{-C} \varepsilon$$

- Problem is ill-posed as $|a_{d_j-1,j}| \to 0$ and/or $\delta \to 0$.
- Numerical simulations confirm qualitative predictions.

D.Batenkov, Y.Yomdin, "On the accuracy of solving confluent Prony systems", *SIAM J.Appl.Math., 73(1), pp.134–154, 2013.*

Decimation

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad {\scriptstyle k=\eta,\eta+\sigma,\eta+2\sigma,\dots,\eta+(C-1)\sigma}$$

$$\delta_{\sigma} = \min_{i<j} |z_i^{\sigma} - z_j^{\sigma}|, \quad |\Delta m_k| < \varepsilon$$

$$\Longrightarrow \quad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta_{\sigma}^{-C} \sigma^{-d_j} \varepsilon$$

- Decimation provides improvement by $\sim \sigma^{d_j}$

D.Batenkov, "Decimated Generalized Prony systems", *submitted.*

Decimation

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad {\scriptstyle k=\eta,\eta+\sigma,\eta+2\sigma,\dots,\eta+(C-1)\sigma}$$

$$\delta_{\sigma} = \min_{i<j} |z_i^{\sigma} - z_j^{\sigma}|, \quad |\Delta m_k| < \varepsilon$$

$$\Longrightarrow \quad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta_{\sigma}^{-C} \sigma^{-d_j} \varepsilon$$

- Decimation provides improvement by $\sim \sigma^{d_j}$
- Superresolution: for $\delta_\sigma \ll 1$ we have $\delta_\sigma \sim \sigma\delta$ and so improvement by $\sim \sigma^{C+d_j}$ (!!).

D.Batenkov, "Decimated Generalized Prony systems", *submitted.*

Decimation

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad {\scriptstyle k=\eta,\eta+\sigma,\eta+2\sigma,\dots,\eta+(C-1)\sigma}$$

$$\delta_{\sigma} = \min_{i<j} |z_i^{\sigma} - z_j^{\sigma}|, \quad |\Delta m_k| < \varepsilon$$

$$\implies \quad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta_{\sigma}^{-C} \sigma^{-d_j} \varepsilon$$

- Decimation provides improvement by $\sim \sigma^{d_j}$
- Superresolution: for $\delta_\sigma \ll 1$ we have $\delta_\sigma \sim \sigma\delta$ and so improvement by $\sim \sigma^{C+d_j}$ (!!).
- Qualitative similarity to CRB and Donoho's lower bounds.

D.Batenkov, "Decimated Generalized Prony systems", *submitted.*

Decimation

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad {\scriptstyle k=\eta,\eta+\sigma,\eta+2\sigma,\dots,\eta+(C-1)\sigma}$$

$$\delta_\sigma = \min_{i<j} |z_i^\sigma - z_j^\sigma|, \quad |\Delta m_k| < \varepsilon$$

$$\Longrightarrow \quad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta_\sigma^{-C} \sigma^{-d_j} \varepsilon$$

- Decimation provides improvement by $\sim \sigma^{d_j}$
- Superresolution: for $\delta_\sigma \ll 1$ we have $\delta_\sigma \sim \sigma\delta$ and so improvement by $\sim \sigma^{C+d_j}$ (!!).
- Qualitative similarity to CRB and Donoho's lower bounds.
- Reduction in number of samples without sacrificing accuracy too much

D.Batenkov, "Decimated Generalized Prony systems", *submitted.*

Decimation

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}, \quad {\scriptstyle k=\eta,\eta+\sigma,\eta+2\sigma,\dots,\eta+(C-1)\sigma}$$

$$\delta_{\sigma} = \min_{i<j} |z_i^{\sigma} - z_j^{\sigma}|, \quad |\Delta m_k| < \varepsilon$$

$$\implies \quad |\Delta z_j| \sim |a_{d_j-1,j}|^{-1} \delta_{\sigma}^{-C} \sigma^{-d_j} \varepsilon$$

- Decimation provides improvement by $\sim \sigma^{d_j}$
- Superresolution: for $\delta_{\sigma} \ll 1$ we have $\delta_{\sigma} \sim \sigma\delta$ and so improvement by $\sim \sigma^{C+d_j}$ (!!).
- Qualitative similarity to CRB and Donoho's lower bounds.
- Reduction in number of samples without sacrificing accuracy too much
- Problem is ill-posed as $|a_{d_j-1,j}| \to 0$ and/or $\delta_{\sigma} \to 0$.

D.Batenkov, "Decimated Generalized Prony systems", *submitted.*

Why this works?

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k = 0, \sigma, 2\sigma, \ldots, (C-1)\sigma$$

$$b_{\ell,j} = a_{\ell,j}\sigma^\ell,\ w_j = z_j^\sigma,\ n_k = m_{k\sigma},\ |w_i - w_j| > \delta_\sigma$$

Why this works?

$$m_k = \sum_{j=1}^{p} z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell, \quad k = 0, \sigma, 2\sigma, \ldots, (C-1)\sigma$$

$$b_{\ell,j} = a_{\ell,j}\sigma^\ell,\ w_j = z_j^\sigma,\ n_k = m_{k\sigma},\ |w_i - w_j| > \delta_\sigma$$

$$n_k = \sum_{j=1}^{p} w_j^k \sum_{\ell=0}^{d_j-1} b_{\ell,j} k^\ell, \quad k = 0, 1, \ldots, C-1$$

$$|\Delta w_j| \sim \frac{1}{|b_{d_j-1,j}|{\delta_\sigma}^C}\varepsilon$$

Why this works?

Rescaling!

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \ldots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \dots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \dots, N$, i.e. $\eta \sim N$, $\sigma = 1$.

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \ldots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \ldots, N$, i.e. $\eta \sim N$, $\sigma = 1$.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1}$.

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \ldots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \ldots, N$, i.e. $\eta \sim N$, $\sigma = 1$.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1}$.
- Batenkov & Yomdin (Math.Comp. 81(2012), pp.277–318): take d above to be half the actual smoothness.

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \ldots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \ldots, N$, i.e. $\eta \sim N$, $\sigma = 1$.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1}$.
- Batenkov & Yomdin (Math.Comp. 81(2012), pp.277–318): take d above to be half the actual smoothness.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-\lfloor \frac{d}{2} \rfloor - 1}$.

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \dots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \dots, N$, i.e. $\eta \sim N$, $\sigma = 1$.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1}$.
- Batenkov & Yomdin (Math.Comp. 81(2012), pp.277–318): take d above to be half the actual smoothness.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-\lfloor \frac{d}{2} \rfloor - 1}$.
- Decimation: take $\eta = \sigma = \left\lfloor \frac{N}{C} \right\rfloor$, this gives improvement by σ^{d+1}

Eckhoff's problem via Prony systems

Eckhoff system

$$(\imath k)^{d+1} c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{\ell} c_{\ell,j}, \quad k = 0, 1, \ldots, N$$

$$|\Delta c_k(\Phi)| \sim k^{-d-2} \Longrightarrow |\Delta LHS| \sim k^{-1}$$

- Eckhoff (1995): Solve with $k = N - C + 1, \ldots, N$, i.e. $\eta \sim N$, $\sigma = 1$.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1}$.
- Batenkov & Yomdin (Math.Comp. 81(2012), pp.277–318): take d above to be half the actual smoothness.
  $\Longrightarrow |\Delta \xi_j| \sim N^{-\lfloor \frac{d}{2} \rfloor - 1}$.
- Decimation: take $\eta = \sigma = \lfloor \frac{N}{C} \rfloor$, this gives improvement by σ^{d+1}
  $\Longrightarrow |\Delta \xi_j| \sim N^{-1} \times N^{-d-1} = N^{-d-2}$.

A-priori bounds

$$f = \underbrace{\Phi}_{\text{piecewise polynomial}} + \underbrace{\Psi}_{\in C^d(S^1)}$$

$$\min_{i<j} |\xi_i - \xi_j| \geqslant J > 0$$

$$|a_{\ell,j}| \leqslant A < \infty$$

$$|a_{0,j}| \geqslant B > 0$$

$$|c_k(\Psi)| \leqslant R \cdot k^{-d-2}$$

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$
2. Localize each ξ_j by multiplying f with a mollifier (convolution in the Fourier domain)

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$
2. Localize each ξ_j by multiplying f with a mollifier (convolution in the Fourier domain)
3. Solve resulting Eckhoff system for one point with decimation: $\sigma = \eta = \lfloor \frac{N}{d+2} \rfloor$

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$
2. Localize each ξ_j by multiplying f with a mollifier (convolution in the Fourier domain)
3. Solve resulting Eckhoff system for one point with decimation: $\sigma = \eta = \left\lfloor \frac{N}{d+2} \right\rfloor$
   - Recovery of $\xi_j = \xi$ boils down to solving a single polynomial equation $P(s) = 0$.

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$
2. Localize each ξ_j by multiplying f with a mollifier (convolution in the Fourier domain)
3. Solve resulting Eckhoff system for one point with decimation: $\sigma = \eta = \left\lfloor \frac{N}{d+2} \right\rfloor$
   - Recovery of $\xi_j = \xi$ boils down to solving a single polynomial equation $P(s) = 0$.
   - Perturbations of the roots of P are explicitly analyzed using Rouche's principle.

Algorithm

$$f = \Phi^{(d)} + \Psi$$

1. Obtain initial approximations for $\{\xi_1, \ldots, \xi_p\}$
2. Localize each ξ_j by multiplying f with a mollifier (convolution in the Fourier domain)
3. Solve resulting Eckhoff system for one point with decimation: $\sigma = \eta = \left\lfloor \frac{N}{d+2} \right\rfloor$
   - Recovery of $\xi_j = \xi$ boils down to solving a single polynomial equation $P(s) = 0$.
   - Perturbations of the roots of P are explicitly analyzed using Rouche's principle.
4. The final approximation is

$$\tilde{f} = \tilde{\Phi}\left(\left\{\tilde{a}_{\ell,j}, \tilde{\xi}_j\right\}\right) + \sum_{|k| \leqslant N} \left\{ c_k(f) - \frac{1}{2\pi} \sum_{j=1}^{p} e^{-\imath \tilde{\xi}_j k} \sum_{\ell=0}^{d} \frac{\tilde{a}_{\ell,j}}{(\imath k)^{\ell+1}} \right\} e^{\imath k x}.$$

Main result

Theorem

The approximation $\tilde{f}$ *satisfies, for* $N \gg 1$:

$$|\xi_j - \tilde{\xi}_j| \sim N^{-d-2},$$
$$|a_{\ell,j} - \tilde{a}_{\ell,j}| \sim N^{\ell-d-1},$$
$$|f(x) - \tilde{f}(x)| \sim N^{-d-1}.$$

D.Batenkov & Y.Yomdin, "Algebraic Fourier reconstruction of piecewise-smooth functions", *Math.Comp. 81(2012), pp.277–318*
D.Batenkov, "Complete algebraic reconstruction of piecewise-smooth functions from Fourier data", *submitted*.

Main result

Theorem

The approximation $\tilde{f}$ satisfies, for $N \gg 1$:

$$\begin{aligned} |\xi_j - \tilde{\xi}_j| &\sim N^{-d-2}, \\ |a_{\ell,j} - \tilde{a}_{\ell,j}| &\sim N^{\ell-d-1}, \\ |f(x) - \tilde{f}(x)| &\sim N^{-d-1}. \end{aligned}$$

- The pointwise bound is valid "away from the jumps"

D.Batenkov & Y.Yomdin, "Algebraic Fourier reconstruction of piecewise-smooth functions", *Math.Comp. 81(2012), pp.277–318*
D.Batenkov, "Complete algebraic reconstruction of piecewise-smooth functions from Fourier data", *submitted*.

Main result

Theorem

The approximation $\tilde{f}$ satisfies, for $N \gg 1$:

$$\begin{aligned} |\xi_j - \tilde{\xi}_j| &\sim N^{-d-2}, \\ |a_{\ell,j} - \tilde{a}_{\ell,j}| &\sim N^{\ell-d-1}, \\ |f(x) - \tilde{f}(x)| &\sim N^{-d-1}. \end{aligned}$$

- The pointwise bound is valid "away from the jumps"
- The constants are fairly explicit, depending on the a-priori bounds A, B, J, R and on the "size" of the problem t, d.

D.Batenkov & Y.Yomdin, "Algebraic Fourier reconstruction of piecewise-smooth functions", *Math.Comp. 81(2012), pp.277–318*
D.Batenkov, "Complete algebraic reconstruction of piecewise-smooth functions from Fourier data", *submitted*.

Construction of the polynomial

$$\begin{aligned} m_k &= z^k \sum_{\ell=0}^{d} c_\ell k^\ell \\ (x-z)^{d+1} &= x^{d+1} + r_d x^d + \cdots + r_0 \\ \Longrightarrow 0 &= m_k r_d + m_{k+1} r_{d-1} + \cdots + m_{k+d} r_0 + m_{k+d+1}, \quad k = 0, 1, \ldots \end{aligned}$$

Example: $d = 1$

$$m_k = z^k(c_0 + kc_1)$$

$$(x - z)^2 = x^2 \underbrace{-2z}_{=r_0} x + \underbrace{z^2}_{=r_1}$$

$$p_k(w) = m_k w^2 - 2wm_{k+1} + m_{k+2}$$

$$k \to \infty : p_k(w) \to c_1 z^k (w - z)^2$$

If p_k is perturbed by $O(k^{-2})$ then as $k \to \infty$ we have

$$|\tilde{z} - z| \sim \sqrt{k^{-2}},$$

i.e. first order accuracy!

Example: $d = 1$ - decimated setting

$$m_k = z^k(c_0 + kc_1)$$
$$q_k(w) = m_k w^{2k} - 2w^k m_{2k} + m_{3k}$$
$$k \to \infty : q_k(w) \to c_1 z^k[w^{2k} - 4z^k w^k + 3z^{2k}]$$

If q_k is perturbed by $O(k^{-2})$ then as $k \to \infty$ we have roots z^k and $3z^k$, and therefore

$$|\tilde{z}^k - z^k| \sim k^{-2} \Longrightarrow |\tilde{z} - z| \sim k^{-3},$$

i.e. full accuracy!

Spectral edge detection: what next?

- Efficient 1D algorithm

Spectral edge detection: what next?

- Efficient 1D algorithm
- Lower bounds for performance

Spectral edge detection: what next?

- Efficient 1D algorithm
- Lower bounds for performance
- Dealing with incorrect model

Spectral edge detection: what next?

- Efficient 1D algorithm
- Lower bounds for performance
- Dealing with incorrect model
- Piecewise-analytic case

Spectral edge detection: what next?

- Efficient 1D algorithm
- Lower bounds for performance
- Dealing with incorrect model
- Piecewise-analytic case
- Application to PDEs?

Prony: global accuracy bounds

- Algebraic methods - potentially best results, not only asymptotic?

Prony: global accuracy bounds

- Algebraic methods - potentially best results, not only asymptotic?
- "Algebraic superresolution"

Prony: global accuracy bounds

- Algebraic methods - potentially best results, not only asymptotic?
- "Algebraic superresolution"
- Study algebraic-geometric structure of the "Prony manifold"

Prony: global accuracy bounds

- Algebraic methods - potentially best results, not only asymptotic?
- "Algebraic superresolution"
- Study algebraic-geometric structure of the "Prony manifold"
- Oversampling?

$\int\int\int$ THANK YOU $\int\int\int$