A symbolic approach to polynomial optimization over basic closed semialgebraic sets

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Aim:

- If $\deg(f_i), \deg(g) \leq d$ and $H(f_i), H(g) \leq H$, obtain explicit bounds $\delta > 0$ and $b > 0$ depending on n, m, d, H such that, if $g_{\min} \neq 0$, then $\deg(g_{\min}) \leq \delta$ and $|g_{\min}| \geq b$.

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Assumption: $E_{\min}$ has a compact connected component.

Related work

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- Numerical or symbolic-numerical computation methods

  Lasserre (2001), Parrilo-Sturmfels (2003), Nie-Demmel-Sturmfels (2006), Schweighofer (2006); Greuet-Guo-Safey El Din-Zhi (2012), among others. Certificates of positivity, SDP, moments.

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- Symbolic computation methods
  - Quantifier elimination over the reals, Basu-Pollack-Roy (1996).
  - Safey El Din (2008), computation of generalized critical values.

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Find a finite set of points containing at least one point in each compact connected component of $E_{\min}$.

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Lagrange multipliers: If g attains a minimum on $E = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_m(x) = 0\}$ at x^*, (under certain assumptions on $g, f_1, \ldots, f_m$) there exists $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

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Difficulties:

- Infinitely many minimizing points.
- Degenerate systems.

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$d \in \mathbb{Z}_{>0}$ an even upper bound for the degrees of g and f_i,
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Since $\hat{f}_i(x) > 0$ for every $1 \le i \le m$:

$$E_0 = E \quad \text{and} \quad E_{t_1} \subset E_{t_2} \text{ if } 0 \le t_1 \le t_2.$$

General strategy

Minimize $G_t = g(x) + t\,\hat{g}(x)$ on E_t for generic t and let $t \to 0$.

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For t small enough, minimizers of G_t on E_t are solutions to a system

$$F_{S,\sigma} : \left\{ F_{i_1}^{\sigma_{i_1}}(t,x) = 0, \dots, F_{i_s}^{\sigma_{i_s}}(t,x) = 0 \right.$$

$$G_{S,\sigma} : \left\{ \nabla_x G(t,x) = \sum_{1 \le j \le s} \lambda_j \nabla_x F_{i_j}^{\sigma_{i_j}}(t,x) \right.$$

for some (S,σ) such that

- $S = \{i_1, \dots, i_s\} \subset \{1, \dots, m\}$ with $s \le n$, and
- $\sigma \in \{+,-\}^S$ with $\sigma_i = +$ for $l + 1 \le i \le m$.

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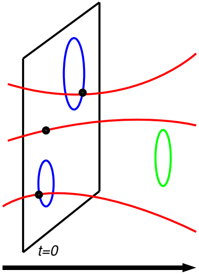

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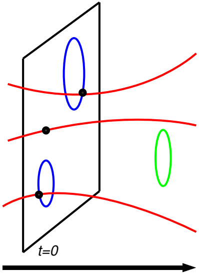

Proposition

Let C be a compact connected component of

$$E = \{f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0\}.$$

Then, there exist $x^* \in C$, $S \subset \{1, \ldots, m\}$ with $0 \leq |S| \leq n$, and $\sigma \in \{+, -\}^S$ with $\sigma_i = +$ for $l + 1 \leq i \leq m$, such that

$$x^* \in \pi_x(V_{S,\sigma} \cap \{t = 0\}) \text{ and } g(x^*) = \min\{g(x) \mid x \in C\}.$$

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Basic result:

Let $Q(u) = \sum_{j=0}^{D} c_j u^j \in \mathbb{Z}[u] \setminus \{0\}$ and $M \in \mathbb{Z}$ such that $|c_j| < M$ for $0 \leq j \leq D$. If $u_0 \in \mathbb{C} \setminus \{0\}$ is a root of Q, then $|u_0| \geq M^{-1}$.

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Let $\nu_{S,\sigma}$ be the largest power of t dividing $R_{S,\sigma}(t,u)$ and

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For each compact connected component C of E, there exists (S,σ) such that $g_{\min,C}$ is a root of $Q_{S,\sigma}$.

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- classical Bézout-type upper bounds for the degree of multihomogeneos resultants,
- upper bounds for heights of sparse resultants (Sombra, 2004).

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Theorem (J.-Perrucci-Tsigaridas)

Let $E = \{f_1(x) = \dots f_l(x) = 0, f_{l+1}(x) \geq 0, \dots, f_m(x) \geq 0\}$ be defined in $\mathbb{R}^n$ by $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ and let $g \in \mathbb{Z}[x_1, \dots, x_n]$. If $\deg(f_i), \deg(g) \leq d$ for an even positive integer d, and $H(f_i), H(g) \leq H$, the minimum value that g takes over a compact connected component C of E is a real algebraic number $g_{\min,C}$ such that, if it is nonzero,

$$\deg(g_{\min,C}) \leq 2^{n-1} d^n \quad \text{and} \quad |g_{\min,C}| \geq (2^{4-\frac{n}{2}} \mathcal{H} d^n)^{-n 2^n d^n}$$

where $\mathcal{H} := \max(H, 2n + 2m)$.

Noncompact situations

- The upper bound for the degree remains valid in the case where C is a non-compact connected component of E, provided that g attains a minimum $g_{\min,C}$ on C.
- If, in addition, the set $C_{\min} = \{x \in C \mid g(x) = g_{\min,C}\}$ has a compact connected component, the lower bound for $|g_{\min,C}|$ also holds.

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In both cases, the bounds follow from the previous result applied to a set which is constructed from E by adding a new constraint of type $M - \sum_{i=1}^{n} x_i^2 \geq 0$ for a suitable $M \in \mathbb{R}_{>0}$.

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- $C_1 \times C_2$ is a connected component of $T_1 \times T_2$,
- $D : \mathbb{R}^{2n} \to \mathbb{R}$, $D(x, y) = \sum_{1 \leq k \leq n} (x_k - y_k)^2$ satisfies $d(C_1, C_2)^2 = \min\{D(x, y) \mid (x, y) \in C_1 \times C_2\}$,
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Theorem (J.-Perrucci-Tsigaridas)

If $\deg(f_i), \deg(g_j) \leq d$ and $H(f_i), H(g_j) \leq H$, then

$$d(C_1, C_2) \geq \left(2^{4-n}\mathcal{H}d^{2n}\right)^{-n4^n d^{2n}}$$

where $\mathcal{H} := \max(H, 4n + 2m_1 + 2m_2)$.

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General strategy: adapt the previous techniques to the algorithmic framework.

- Apply a symbolic deformation based on the Newton-Hensel lifting to compute a finite set of points containing minimizers.
- Compare the values that the function g takes on the computed points using Thom encodings by analyzing sign conditions on suitable families of univariate polynomials.

A modified deformation

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- Let $t \to 1$ to obtain points in $E_{\min}$.

$\mathcal{S} = \{(S, \sigma) \mid S \subset \{1, \dots, m\}, 0 \le |S| \le n, \text{ and } \sigma \in \{+, -\}^S,$
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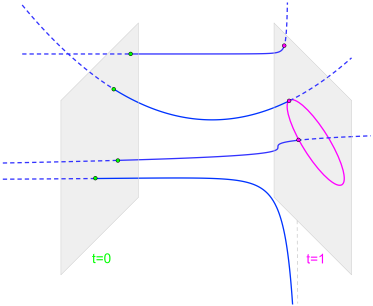

- For generic t, $(F_{S,\sigma}, G_{S,\sigma})$ has finitely many solutions.
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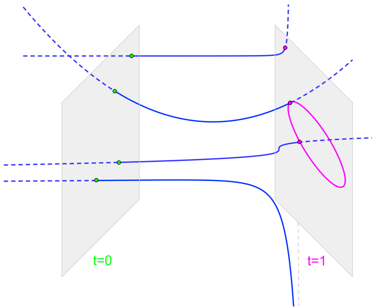

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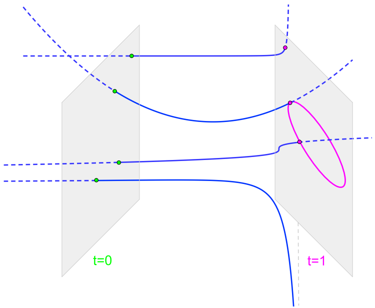

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The set $\displaystyle\bigcup_{(S,\sigma)\in\mathcal{S}} \pi_x(V_{S,\sigma} \cap \{t = 1\})$ is finite and contains a point in every compact connected component of $E_{\min}$.

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- For every $(S, \sigma) \in \mathcal{S}$, compute a finite set $\mathcal{P}_{S,\sigma}$ containing $\pi_x(V_{S,\sigma} \cap \{t = 1\})$.
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- Compare the minimum values that g takes on the sets $\mathcal{P}_{S,\sigma}\cap E$ for different (S,σ).
  Subroutine: `ComparingMinimums`

Representing finite sets

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The real roots of p correspond to the real points of V.

Geometric resolution of critical points

Assume $S = \{1, \dots, s\}$ with $s \leq n$ and $\sigma = \{+\}^S$. Recall that

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Compute a geometric resolution of $\pi_x(V_{S,\sigma} \cap \{t = 1\})$ going from $t = 0$ to $t = 1$.

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Complexity: $O_{\log}(n^3 D_s^2 (L + dn + n^3))$

- $D_s = \binom{n}{s} d^s (d-1)^{n-s}$
- L = length of a straight-line program encoding $f_1, \dots, f_s, g$

Thom encoding of real algebraic numbers

For $p \in \mathbb{Q}[u]$ and $\xi \in \mathbb{R}$ such that $p(\xi) = 0$, the Thom encoding of ξ as a root of p is the sequence $(\mathrm{sign}(p'(\xi)), \dots, \mathrm{sign}(p^{(\deg p)}(\xi)))$, where $\mathrm{sign} : \mathbb{R} \to \{-1, 0, 1\}$.

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- Given the Thom encodings $(\tau_{1,1}, \dots, \tau_{1,\deg p})$ and $(\tau_{2,1}, \dots, \tau_{2,\deg p})$ of two different real roots ξ_1 and ξ_2 of p, it is possible to decide which is the smallest between ξ_1 and ξ_2: if $k_0 = \max\{k \mid \tau_{1,k} \neq \tau_{2,k}\}$, then
  - if $\tau_{1,k_0+1} = \tau_{1,k_0+1} = 1$, we have $\xi_1 < \xi_2 \iff \tau_{1,k_0} < \tau_{2,k_0}$
  - if $\tau_{1,k_0+1} = \tau_{1,k_0+1} = -1$, we have $\xi_1 < \xi_2 \iff \tau_{1,k_0} > \tau_{2,k_0}$

Sign conditions for univariate polynomials

A realizable sign condition for polynomials $h_1, \ldots, h_m \in \mathbb{R}[u]$ is $\sigma = (\sigma_1, \ldots, \sigma_m) \in \{<, =, >\}^m$ such that

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- Computing the Thom encodings of the real roots of $p \in \mathbb{R}[u]$ amounts to computing the realizable sign conditions for $p, p', \ldots, p^{(\deg p)}$ where p vanishes.

Computing minimizers with Thom encodings

$\mathcal{P} \subset \mathbb{C}^n$ finite set given by a geometric resolution $(p, v_1, \dots, v_n)$

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1. Determine whether $\mathcal{P} \cap E \neq \emptyset$ by computing the list of realizable sign conditions for $f_1(v(u)), \ldots, f_m(v(u))$ over the real roots of p, where $v = (v_1, \ldots, v_n)$.

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Computing minimizers with Thom encodings

$\mathcal{P} \subset \mathbb{C}^n$ finite set given by a geometric resolution $(p, v_1, \ldots, v_n)$

$$E = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, f_{l+1}(x) \geq 0, \ldots, f_m(x) \geq 0\}$$

1. Determine whether $\mathcal{P} \cap E \neq \emptyset$ by computing the list of realizable sign conditions for $f_1(v(u)), \ldots, f_m(v(u))$ over the real roots of p, where $v = (v_1, \ldots, v_n)$.
2. If $\mathcal{P} \cap E \neq \emptyset$:
   - Let $h(u) = \mathrm{Res}_{\tilde{u}}(p(\tilde{u}), u - g(v(\tilde{u})))$,
   - Compute the list of realizable sign conditions for $f_1(v(u)), \ldots, f_m(v(u))$, $p'(u), \ldots p^{(\deg p - 1)}(u)$, $h'(g(v(u))), \ldots, h^{(\deg p-1)}(g(v(u)))$ over the real roots of p,
   - Go through this list to find the Thom encodings of minimizers for g on $\mathcal{P} \cap E$.

Comparing values of g

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4. Go through this list to find the Thom encodings of the minimizers for g on $\mathcal{P}_1 \cup \mathcal{P}_2$.

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- $g \in \mathbb{Q}[x_1, \ldots, x_n]$ attaining a minimum value at E in a set $E_{\mathrm{min}} \neq \emptyset$ with at least one compact connected component.

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There is a probabilistic procedure that computes a finite family $\big\{\big((p_i, v_{i,1}, \ldots, v_{i,n}), \tau_i\big)\big\}_{i \in \mathcal{I}}$ where, for every $i \in \mathcal{I}$,

- $(p_i, v_{i,1}, \ldots, v_{i,n})$ is a geometric resolution and
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Complexity: $O_{\log}\big((n^3(L + dn + n^3)D^2 + (m + D)D^2)\Upsilon\big)$

- $d \geq \deg(f_i), \deg(g)$ an even integer,
- L = length of an slp encoding $f_1, \ldots, f_m, g$,
- $D = \max\limits_{0 \leq s \leq \min\{n,m\}} \binom{n}{s} d^s (d-1)^{n-s}$, and $\Upsilon \leq \sum\limits_{0 \leq s \leq \min\{n,m\}} \binom{m}{s} 2^s$.

Thank you for your attention!