Exponential transforms, resultants and moments

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Overview

Abstract

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The Taylor expansion at infinity of the Cauchy transform of a domain in the complex plane has the harmonic moments as coefficients. Similarly, there is a double Cauchy transform (in two complex variables) having the complete two index set of complex moments as coefficients. Assuming these moments are known, but the domain itself is unknown, one can in principle obtain a good defining function for the boundary by taking the exponential of this double Cauchy transform. The resulting function is called the exponential transform, and it has many remarkable properties. In particular, the above procedure for determining the unknown boundary turns out to be effective in many cases.
The best cases have some algebraic flavor, and then the exponential transform also become an algebraic object (actually a rational function), closely related to a certain meromorphic resultant. I plan to review some of these matters, but I hope also to come up with some new results. The talk is based on joint work with Mihai Putinar, Vladimir Tkachev and others.

New abstract

Abstract

The main topic will be potential theoretic skeletons for polyhedra/polygons and other types of bodies.
There is an approach based on asymptotic distribution of zeros of orthogonal polynomials, or eigenvalues of associated matrices. In this context there is a new kind of polynomials, based on an exponential transform, whose zeros seem to have a tendency to go deeply into the domain in question, hence seem to be attracted by some kind of skeleton, but whose behavior we still cannot explain by some theory. Part of the talk will touch on this topic, and in general the talk represents past and ongoing joint work with Mihai Putinar and Nikos Stylianopoulos.

Cauchy and exponential transforms I

Definition

Cauchy transform of a domain $\Omega \subset \mathbb{C}$:

$$C_\Omega(z) = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Double Cauchy transform:

$$C_\Omega(z, w) = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}$$

Exponential transform:

$$E_\Omega(z, w) = \exp C_\Omega(z, w)$$

These transforms are defined in all $\mathbb{C}$ and are analytic/antianalytic outside Ω

Cauchy and exponential transforms II

In addition we have the *interior exponential transform*

$$H(z,w) = -\frac{\partial^2 E(z,w)}{\partial\bar{z}\partial w},$$

which is analytic/antianalytic in Ω. Since $1 - E(z,w) \to 0$ as $z \to \infty$, or $w \to \infty$,

$$1 - E(z,w) = \frac{1}{\pi^2}\int_\Omega\int_\Omega H(u,v)\frac{dm(u)}{u-z}\frac{dm(v)}{\bar{v}-\bar{w}}.$$

Theorem

- *$H(z,w)$ is positive definite (in Ω)*
- *$1 - E(z,w)$ is positive semidefinite outside Ω*

The Schwarz function

The Cauchy integral

$$D(z) = -\frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{\bar{\zeta} d\zeta}{\zeta - z} = \begin{cases} C_{\Omega}(z) & \text{when } z \in \mathbb{C} \setminus \overline{\Omega}, \\ C_{\mathbb{C}\setminus\Omega}(z) & \text{when } z \in \Omega \end{cases}$$

makes the jump $\bar{z}$ across $\partial\Omega$, and if $\partial\Omega$ is analytic (henceforth assumed) the two functions $D_{\text{outside}}(z)$ and $D_{\text{inside}}(z)$, defined on either sides of $\partial\Omega$, have analytic continuations across this boundary. Hence the difference

$$S(z) = D_{\text{outside}}(z) - D_{\text{inside}}(z)$$

is analytic in a neighborhood of $\partial\Omega$ and satisfies

$$S(z) = \bar{z} \quad (z \in \partial\Omega).$$

This is the *Schwarz function* of $\partial\Omega$.

Definitions of moments

Definition

- *Complex moments*:

$$M_{kj} = \frac{1}{\pi}\int_{\Omega} z^k \bar{z}^j \, dm(z) \quad (k, j \geq 0)$$

- *Harmonic moments*:

$$M_k = M_{k0} = \frac{1}{2\pi \mathrm{i}}\int_{\partial\Omega} z^k \bar{z} dz \quad (\text{make sense for all } k \in \mathbb{Z})$$

- *Exponential moments*:

$$B_{kj} = \frac{1}{\pi^2}\int_{\Omega}\int_{\Omega} H(z,w) z^k \bar{w}^j dm(z) dm(w) \quad (k, j \geq 0)$$

Generators of moments

The previous transforms generate the moments as follows:

$$C_\Omega(z) = \sum_{k=0}^{\infty} \frac{M_k}{z^{k+1}},$$

$$C_\Omega(z,w) = -\sum_{k,j\geq 0} \frac{M_{kj}}{z^{k+1}\,\bar{w}^{j+1}},$$

$$1 - E_\Omega(z,w) = \sum_{k,j\geq 0} \frac{B_{kj}}{z^{k+1}\,\bar{w}^{j+1}},$$

$$S(z) = \sum_{k=-\infty}^{\infty} \frac{M_k}{z^{k+1}}.$$

Since $E_\Omega(z,w) = \exp C_\Omega(z,w)$ the exponential moments can be obtained directly from the complex moments by

$$1 - \exp[-\sum_{k,j\geq 0} \frac{M_{kj}}{z^{k+1}\bar{w}^{j+1}}] = \sum_{k,j\geq 0} \frac{B_{kj}}{z^{k+1}\bar{w}^{j+1}}.$$

The coefficient matrices have the following positivity properties:

- $(M_{kj})_{0\leq k,j\leq N}$ is positive definite for any N (obvious from definition)
- $(B_{kj})_{0\leq k,j\leq N}$ is positive semi-definite for any N (consequence of a previous theorem)

Therefore they can be used as Gram-matrices to generate orthogonal polynomials, in $L^2(\Omega)$, and in a suitable Hilbert space $\mathcal{H}(\Omega)$:

$$M_{kj} = < z^k, z^j >_{L^2(\Omega)}$$

$$B_{kj} = < z^k, z^j >_{\mathcal{H}(\Omega)}$$

The logarithmic potential

For a general measure μ we set

$$U_\mu(z) = \frac{1}{2\pi}\int \log\frac{1}{|z-\zeta|}d\mu(\zeta) \quad (\textit{logarithmic potential})$$

$$C_\mu(z) = -\frac{1}{\pi}\int \frac{d\mu(\zeta)}{\zeta - z} \quad (\textit{Cauchy transform}).$$

Then

$$\frac{\partial U_\mu(z)}{\partial z} = -\frac{1}{4}C_\mu(z),$$

$$\mu = \frac{\partial C_\mu}{\partial \bar{z}} = -\Delta U_\mu.$$

When $\mu = \chi_\Omega m$ we write $U_\Omega = U_\mu$, $C_\Omega = C_\mu$.

Potential theoretic skeletons (“mother bodies”)

Definition

Let Ω be a bounded domain in $\mathbb{C}$ or, more generally, in $\mathbb{R}^n$ and consider Ω as a body of density one. A measure μ is a *potential theoretic skeleton*, or *mother body*, for Ω if

1) $U_\mu = U_\Omega$ outside Ω
2) $U_\mu \geq U_\Omega$ everywhere
3) $\mu \geq 0$
4) $m(\operatorname{supp}\mu) = 0$
5) $\Omega \setminus \operatorname{supp}\mu$ has no relatively compact components.

Remarks “mother bodies”

Remark

a) *The terminology “mother body” goes back to the Bulgarian geophysicist D. Zidarov.*

b) *Conditions 1) and 2) guarantee that Ω can be recovered from μ by a process called partial balayage. Condition 1) alone is not enough.*

c) *If $\Omega \subset \mathbb{C}$ is simply connected, condition 1) is equivalent to*

$$C_\mu = C_\Omega \quad \textit{outside } \Omega$$

and 5) just means that $\mathbb{C} \setminus \operatorname{supp} \mu$ is connected.

d) *In general, very few bodies Ω admit mother bodies at all, and when they exist they need not be unique.*

e) *Polyhedra is an interesting subclass of domains in the context of mother bodies.*

Remarks “mother bodies”

Remark

f) *If one is willing to ignore the conditions 2) and 3) in the requirements of a mother body (that $U_\mu \geq U_\Omega$ and $\mu \geq 0$) then the search for a mother body for a simply connected domain Ω reduces to finding an analytic continuation of the exterior Cauchy transform*

$$D_{outside}(z) = C_\Omega(z) \quad (z \in \mathbb{C} \setminus \overline{\Omega}),$$

or, equivalently, of the Schwarz function

$$S(z) = D_{outside}(z) - D_{inside}(z),$$

to Ω minus a closed nullset which does not disconnect $\mathbb{C}$. In many typical cases this set will consist of finitely many isolated points and/or branch cuts for $S(z)$.

Mother bodies of polyhedra, known results

Remark

a) *Convex polyhedra have unique mother bodies (in any number of dimensions)*
b) *In two dimensions, also non-convex polyhedra have mother bodies, but they are not unique.*
c) *For non-convex polyhedra in higher dimensions very little is known.*
d) *For non-convex polyhedra one sometimes get more natural looking mother bodies by relaxing the condition $\mu \geq 0$, allowing signed measures.*

Some general methods for construction of mother bodies

a) By direct construction (using e.g. Schwarz functions/potentials)
b) By solving a suction version of a Hele-Shaw problem (Laplacian growth in the ill-posed direction)
c) By a version of internal DLA
d) Using a “restoring sandpile” algorithm
e) As asymptotic distribution of zeros of orthogonal polynomials
f) As asymptotic distribution of eigenvalues of matrices (for example random matrices, or matrices obtained as truncations of infinite shift operators)

Direct construction of mother body

If Ω has a mother body μ, then the function

$$u = U_\mu - U_\Omega$$

satisfies

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \setminus \operatorname{supp}\mu, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, if we are seeking a mother body for a domain Ω we may start by trying to find local solutions of the above Cauchy problem (assuming that $\operatorname{supp}\mu$ does not reach $\partial\Omega$) and then try to extend these solutions as far into Ω as possible. There can be no solution in all of Ω, but the places where one has to give up, or has to match different local branches, will be the location for the support of μ. Once u is constructed, μ is obtained as $\mu = 1 - \Delta u$.

Example: Polyhedra

For a polyhedron, $\partial\Omega$ is piecewise flat, and in a neighborhood of each flat piece the local solution of the above Cauchy problem is

$$u(x) = \frac{1}{2}\mathrm{dist}\,(x, \partial\Omega)^2,$$

in any number of dimensions.
In case the polyhedron is convex, and so the intersection of finitely many half-spaces, these branches of u can be glued by simply taking the global minimum of them all. The above formula will then be the global definition of $u(x)$. This gives a mother body, which turns out to be unique.

Example: Semidisk

The above procedure can also be used in more general cases with piecewise analytic boundary. One example is the semidisk

$$\Omega = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, |z| < 1\}.$$

Here one gets

$$u = \min\{u_1, u_2\},$$

where

$$u_1(z) = \frac{1}{2}x^2, \quad u_2(z) = \frac{1}{4}(|z|^2 - 1 - 2\log|z|).$$

This gives a unique mother body, with support on the curve where $u_1 = u_2$.

Squeezing version of Hele-Shaw problem

For any domain Ω, let $p = p_\Omega$ be the unique solution of

$$\begin{aligned} -\Delta p &= 1 \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

In the squeezing Hele-Shaw problem (or Laplacian growth with uniform source distribution) one asks, given $\Omega = \Omega(0)$ for a family of domains $\{\Omega(t)\}$, ($t =$time) such that the normal velocity of $\partial\Omega(t)$ is

$$V_n = -\frac{\partial p}{\partial n}.$$

If h is a fixed harmonic function (defined wherever needed) one then has

$$\begin{aligned} \frac{d}{dt}\left(e^{-t}\int_{\Omega(t)} h\, dm\right) &= -e^{-t}\int_{\Omega(t)} h\, dm - e^{-t}\int_{\partial\Omega(t)} h\frac{\partial p}{\partial n}\, ds \\ &= -e^{-t}\int_{\Omega(t)} h\, dm - e^{-t}\int_{\Omega(t)} h\Delta p\, dm = 0. \end{aligned}$$

Squeezing version of Hele-Shaw problem II

Thus the measures

$$\mu(t) = e^{-t}\chi_{\Omega(t)} m$$

are gravi-equivalent in the sense that

$$\int h \, d\mu(t) = \text{ independent of } t$$

for any fixed harmonic function h (for example $h(z) = \log|z - a|$, $|a| >> 1$). Assuming that the solution $\{\Omega(t)\}$ exists for all $-\infty < t < \infty$ it follows that, as $t \to -\infty$,

$$\mu(t) \rightharpoonup \mu$$

for some measure μ. If the domains $\{\Omega(t)\}$ do not undergo any topological changes under the evolution, then this μ will have all the properties required by a mother body of $\Omega = \Omega(0)$.

Example: An ellipse

Example

Let Ω be the ellipse in $\mathbb{R}^2$ given in terms of the coordinates x and y by

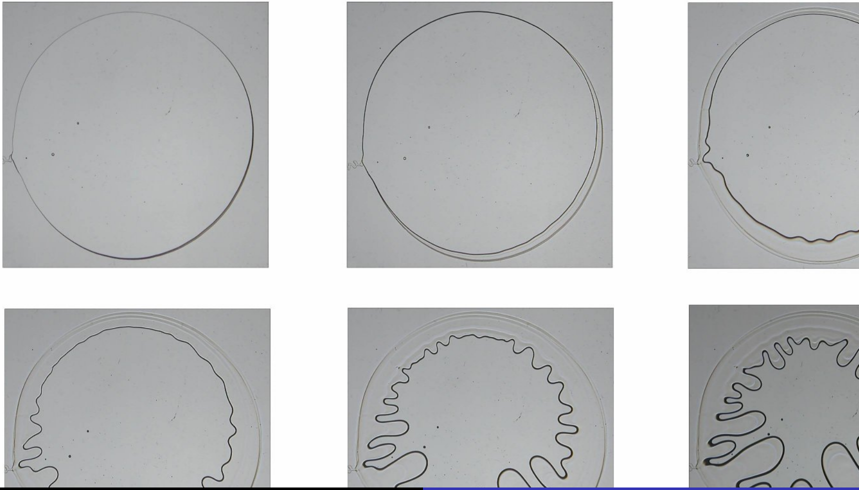
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1,$$

where $a > b > 0$. Then the measure μ on the focal segment $[-c, c]$ ($c = \sqrt{a^2 - b^2}$) defined by

$$d\mu = \frac{2ab}{c^2}\sqrt{c^2 - x^2}dx \qquad (-c < x < c)$$

is a mother body for Ω, and it is unique.
The corresponding squeezed domains $\Omega(t)$, $-\infty < t < \infty$, make up the family of all ellipses confocal to $\Omega = \Omega(0)$.

Pictures of squeezing Hele-Shaw I



Pictures of squeezing Hele-Shaw II

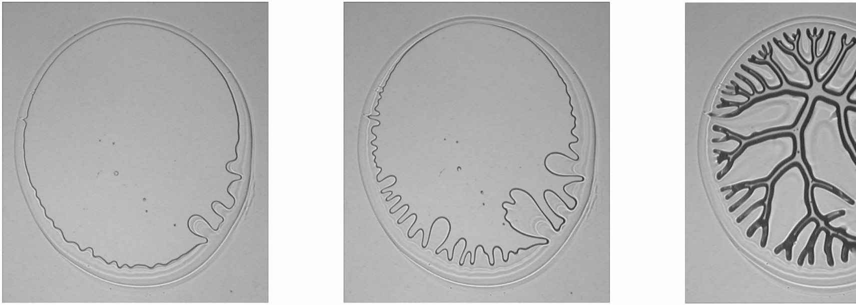


Figure: Performance of Hele-Shaw experiment.

Internal DLA

This approach represents ongoing research of a doctoral student (J. Roos), and I refrain from giving too many details.

External DLA is well-known, with Brownian motion particles emitted at infinity and eventually adding up on a growing crystal like cluster started up at the origin.

Reversing the geometry one gets internal DLA, one version of which has been studied by L. Levine and Y. Peres (Scaling limits for internal aggregation models with multiple sources, J. Analyse (2010)).

Restoring sandpile

Here we just refer to a scheme in a book of D. Zidarov (Inverse Gravimetric Problem in Geoprospecting and Geodesy, 1990 (for English edition)). See next frame.
The algorithm has the flavor of a finite game, and the problem is actually known in game theory (P. Diaconis, W. Fulton: A growth model, a game, an algebra, Lagrange inversion, and characteristic classes, Rend. Sem. Mat. Univ. Pol. Torino 49 (1991), 95-119).

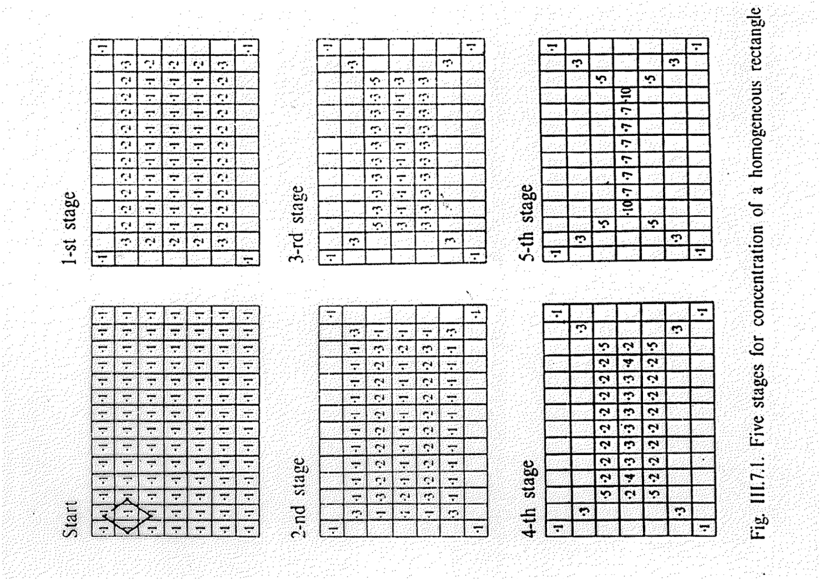


Fig. III.7.1. Five stages for concentration of a homogeneous rectangle

Orthogonal polynomials

Classical families of polynomials include those of

- Fekete
- Chebyshev
- Faber
- Szegö
- Bergman

The counting measures $\nu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$ of the associated zeros $z_1, \ldots, z_n$ of such polynomials $P_n(z) = z^n + \cdots = \prod_{k=1}^{n}(z - z_k)$ generally speaking converge weak* to the equilibrium measure μ_{eq} of the domain in question, or sometimes to a potential theoretic skeleton for μ_{eq}.

Orthogonal polynomials, cont.

More recent families include

- weighted Bergman polynomials arising in random matrix theory
- polynomials defined in terms of the exponential transform

Equilibrium measure

Let

$$g_{\mathbb{C}\setminus\overline{\Omega}}(z,\infty) = \frac{1}{2\pi}\log|z| + \text{harmonic}$$

be the Green's function of the exterior domain. The *equilibrium measure* $\mu_{\rm eq}$ for $\overline{\Omega}$ is the following probability measure, sitting on $\partial\Omega$:

$$d\mu_{\rm eq} = -\frac{\partial g_{\mathbb{C}\setminus\overline{\Omega}}(\cdot,\infty)}{\partial n}\, ds.$$

A notion of potential theoretic skeleton μ for $\mu_{\rm eq}$ may be defined in a similar way as that of a mother body for $\chi_\Omega\, m$. In the case of a polygon it is often made up of *curved* line segments, and so has been named a "*Madonna body*". It may obtained as

$$\mu = \Delta h,$$

where h is a harmonic continuation of $g_{\mathbb{C}\setminus\overline{\Omega}}(\cdot,\infty)$ to $\mathbb{C}$ minus a closed nullset that does not disconnect $\mathbb{C}$ (this nullset then being *supp* μ).

Equilibrium measure, cont.

Recall that if $\partial\Omega$ is analytic, the anti-conformal reflection in $\partial\Omega$ is given in terms of the Schwarz function as

$$z \mapsto \overline{S(z)}.$$

If $\partial\Omega$ is only piecewise analytic, like in the case of a polygon, there are several branches of $S(z)$ and one may typically obtain the harmonic continuation of $g_{\mathbb{C}\setminus\overline{\Omega}}(z, \infty)$ as

$$h(z) = -\inf_{\text{branches}} g_{\mathbb{C}\setminus\overline{\Omega}}(\overline{S(z)}, \infty).$$

However, there may be restrictions of which branches are allowed.

Equilibrium measure, cont.

Example

Assume that the interior conformal map $\varphi : \Omega \to \mathbb{D}$ extends analytically across $\partial\Omega$. Then the Schwarz function can be written as

$$S(z) = \varphi^{-1}\left(\frac{1}{\overline{\varphi(z)}}\right)$$

for $z \in \Omega$ (close to $\partial\Omega$). But φ^{-1} is not necessarily single-valued (outside $\mathbb{D}$), so the above $S(z)$ may have several branches.

The above applies if Ω is a rectangle, and gives a Madonna body in this case. But it does not apply if Ω is a pentagon, because then φ has no (single-valued) analytic continuation across $\partial\Omega$. Still $S(z)$ exists as a multi-valued function, and it gives a harmonic continuation h and a Madonna body, as on the previous slide.

Equilibrium measure, cont.

In the theory of Bergman (and Szegö) polynomials it turns out that having a single-valued continuation of φ is necessary for the zeros to converge to some Madonna body sitting inside Ω.
The Bergman kernel of Ω is given in terms of a conformal map $\varphi : \Omega \to \mathbb{D}$ by

$$K(z,a) = \frac{\varphi'(z)\overline{\varphi'(a)}}{\pi(1-\varphi(z)\overline{\varphi(a)})^2}.$$

Assuming that φ extends beyond $\partial\Omega$, $K(z,a)$ extends outside Ω up to points z where $\varphi(z) = 1/\overline{\varphi(a)}$.

Comparison mother - Madonna

Example

For the ellipse with $c = 1$ the unique mother body is

$$d\mu_{\text{mother}} = \text{const.}\sqrt{1 - x^2}\, dx \qquad (-1 < x < 1),$$

while the Madonna body is

$$d\mu_{\text{Madonna}} = \text{const.}\frac{dx}{\sqrt{1 - x^2}} \qquad (-1 < x < 1).$$

In fact,

$$g_{\mathbb{C}\setminus\overline{\Omega}}(z, \infty) = \frac{1}{2\pi} \log|\zeta| = \frac{1}{2\pi} \log|z - \sqrt{z^2 - 1}| + \text{constant}$$

where $z = \frac{1}{2}(R\zeta + \frac{1}{R\zeta})$ (exterior conformal map) for a suitable $R > 0$. The harmonic continuation h is given by the same expression, and the density of μ_{Madonna} equals the jump of $\frac{\partial h}{\partial y}$ over the focal segment.

Weighted equilibrium measure

The equilibrium measure μ_{eq} is characterized by

$$U_{\mu_{\text{eq}}} = \begin{cases} \gamma = \text{“Robin constant”} & \text{a.e. on } \overline{\Omega}, \\ \gamma - g_{\mathbb{C}\setminus\overline{\Omega}}(\cdot, \infty) \geq \gamma & \text{in } \mathbb{C} \setminus \overline{\Omega}. \end{cases}$$

If Q is an exterior potential, there is also a corresponding *weighted equilibrium measure* μ_Q, characterized by

$$U_{\mu_Q} + Q = \begin{cases} \gamma = \text{a Robin constant} & \text{a.e. on } \operatorname{supp} \mu_Q, \\ \geq \gamma & \text{in } \mathbb{C}. \end{cases}$$

Typical choices for Q are

$$Q(z) = |z|^2 + \operatorname{Re} \sum_{k=2}^{m} t_k z^k, \quad \text{or}$$

$$Q(z) = |z|^2 + U_\sigma(z), \quad \sigma \text{ a measure with compact support.}$$

Weighted quilibrium measures, cont.

It is known (Elbau, Felder, Balogh, Harnard, Hedenmalm, Makarov,…) that also weighted equilibrium measures, and potential theoretic skeletons for them, attract zeros of orthogonal polynomials, namely for weighted Bergman spaces with inner products of the form

$$(f, g) = \int_{\mathbb{C}} f\bar{g}e^{-NQ}\, dm.$$

The polynomials then are of the form

$$P_{n,N}(z) = z^n + \cdots = \prod_{k=1}^{n}(z - z_{k,N})$$

and one considers the “scaling´´ limit as

$$n \to \infty, \quad N \to \infty, \quad \frac{N}{n} \to \text{finite constant.}$$

($1/N$ has the interpretation of being Planck’s constant in some physical applications.)

Weighted equilibrium measures, cont.

The advantage with weighted equilibrium measures, if one is interested in for example mother bodies for polygons, is that they often are absolutely continuous with respect to Lebesgue measure, more precisely of the form

$$d\mu_Q = \Delta Q\, \chi_S\, dm,$$

where $S = \operatorname{supp} \mu_Q$.

Question: Given a domain Ω, is it possible to choose Q so that $\mu_Q = \chi_\Omega m$?

Answer: Yes, just take $Q = -U_\Omega$.

The zeros $\{z_{1,N}, \ldots, z_{n,N}\}$ then are obtained by minimizing

$$Min_{\{z_{1,N},\ldots,z_{n,N}\}} \int_{\mathbb{C}} |\prod_{k=1}^{n}(z - z_{k,N})|^2 e^{-NU_\Omega(z)}\, dm(z)$$

and their counting measures $\nu_{n,N}$ converge weak* to μ_Q, or some skeleton for it (there is some numerical evidence for this kind of behavior, but it seems that the theory is not complete).

Exponential polynomials

With a direct approach, the zeros $\{z_1, \ldots, z_n\}$ are obtained from solving

$$Min_{\{z_1,\ldots,z_n\}} \int_\Omega \int_\Omega H(u,v) \prod_{k=1}^{n} (u - z_k)\overline{(v - z_k)} dm(u) dm(v).$$

They show up an interesting behavior, which is however far from understood at present. (Work in progress.)