Computation of (moment) invariants

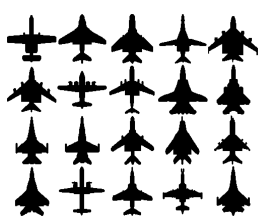
Evelyne Hubert
INRIA Méditerranée

IMS 2014, Singapore

Based on some joint works with Irina Kogan or George Labahn.

1 Moments in pattern recognition

Moments of an image

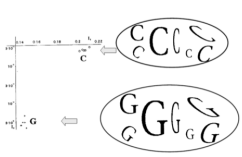


$$\begin{aligned} m_{kl} &= \int_0^1 \int_0^1 x^k y^l \underbrace{f(x,y)}_{grey\,level} \, dxdy \\ &= \sum_{(x,y)_{\text{pixel}}} x^k y^k \overbrace{f(x,y)}^{finite\,values} \end{aligned}$$

Lower order moments are used to discriminate among finite set of shapes.

See Moments & Moment Invariants in Pattern Recognition, by J. Flusser, T. Suk and B. Zitová, Wiley 2009

Moments under geometric transformations



$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \tilde{m}_{20} \\ \tilde{m}_{11} \\ \tilde{m}_{02} \end{bmatrix} = (ac-bd)^2 \begin{bmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{bmatrix} \begin{bmatrix} m_{20} \\ m_{11} \\ m_{02} \end{bmatrix}$$

$$I_1(m) = \frac{m_{20}m_{02} - m_{11}^2}{m_{00}^4}$$

$$I_2(m) = \frac{m_{30}^2 m_{03}^2 + \ldots - 3m_{21}^2 m_{12}^2}{m_{00}^{10}}$$

$$I(\tilde{m}) = I(m)$$

[Hu, Trans. Information Theory (1962)]

2 Rational actions of an algebraic group action

Affine algebraic group $\mathcal{G}$ and action $\star$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\qquad$ $G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

Group structure: $\qquad$ $1 \in \mathcal{G} \quad 1 \cdot \lambda = \lambda \cdot 1 = \lambda$

$$m: \begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \to & \mathcal{G} \\ (\lambda, \mu) & \mapsto & \lambda \cdot \mu \end{array} \qquad \text{and} \qquad i: \begin{array}{ccc} \mathcal{G} & \to & \mathcal{G} \\ \lambda & \mapsto & \lambda^{-1} \end{array}$$

$\lambda \cdot \mu \in \mathbb{K}[\lambda, \mu] \qquad \lambda^{-1} \in \mathbb{K}[\lambda]$

Action of $\mathcal{G}$ on $\mathcal{Z}$ $\qquad$ $\mathcal{Z} = \mathbb{K}^n$

$$\star: \begin{array}{ccc} \mathcal{G} \times \mathcal{Z} & \to & \mathcal{Z} \\ (\lambda, z) & \mapsto & \lambda \star z \end{array}$$

$$1 \star z = z$$

$$(\lambda \cdot \mu) \star z = \lambda \star (\mu \star z)$$

Orbit of $z \in \mathcal{Z}$ $\qquad$ $\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$

e.g. Linear Algebraic Groups

$\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n \times n} \mid \det A = 1\}$

$\mathrm{O}_n(\mathbb{K}) = \left\{A \in \mathbb{K}^{n \times n} \mid A\,{}^tA = I_n\right\}$

$\mathrm{SO}_n(\mathbb{K}) = \mathrm{O}_n(\mathbb{K}) \cap \mathrm{SL}_n(\mathbb{K})$

$\mathrm{GL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n \times n} \mid \det A \neq 0\}$

naturally act on $\mathbb{K}^n$

Classical invariant theory

The action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}[x, y]_d$, the *forms* of degree d

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star p\,(x,\, y) = p(a\,x + b\,y,\, c\,x + d\,y).$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star (z_0\,x^2 + z_1\,xy + z_2\,y^2) = z_0\,(a\,x + b\,y)^2 + \ldots$$

$$= (a^2\,z_0 + ac\,z_1 + c^2\,z_2)\,x^2 + (2ab\,z_0 + (bc + ad)\,z_1 + 2cd\,z_2)\,xy + \ldots$$

$$\rho: \begin{array}{ccc} \mathrm{SL}_2 & \to & \mathrm{GL}_3(\mathbb{K}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad + bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \end{array}$$

Rational actions

$\mathcal{Z} = \mathbb{K}^n$ or an irreducible variety $\qquad \rho : \mathcal{G} \to \mathrm{Bir}(\mathcal{Z})$

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$$

$$q, p_1, \ldots, p_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$$

Möbius transformations: $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star z = \frac{a\,z + b}{c\,z + d}$$

Rational invariants

$\star : \ \mathcal{G} \times \mathcal{Z} \ \to \ \mathcal{Z} \qquad \mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$

Rational invariant: $f \in \mathbb{K}(\mathcal{Z})$ s.t. $f(\lambda \star z) = f(z), \ \forall \lambda \in \mathcal{G}$

Field of rational invariants: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ $\qquad$ finitely generated

THM: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}} = \mathbb{K}(g_1, \ldots, g_k) \ \Leftrightarrow \ \{g_1, \ldots, g_k\}$ separating

[Rosenlicht 56]

Separation property: $\qquad$ For $z, z' \in \mathcal{Z} \setminus \mathcal{W}$

$$z' \in \mathcal{O}_z \quad \Leftrightarrow \quad g_1(z) = g_1(z'), \ \ldots, \ g_k(z) = g_k(z')$$

2.1 Generic orbit ideal

The ideal O of a generic orbit

$$\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$$

$O = (G + (\, Z - \lambda \star z \,)) \cap \mathbb{K}(z)[Z] \qquad \mathcal{V}(O_{|z}) = \bar{\mathcal{O}}_z$

$$\lambda \star z = (p_1(\lambda, z), \ \ldots, \ p_n(\lambda, z)) \in \mathbb{K}[\lambda, z]^n$$

$$(Z - \lambda \star z) = (Z_1 - p_1(\lambda, z), \ldots, Z_n - p_n(\lambda, z)) \subset \mathbb{K}(z)[\lambda, Z]$$

$$\mathcal{O}_z = \mathcal{O}_{\lambda \star z}$$

Prop: $Q(z)$ a canonical representative of $O \ \Rightarrow \ Q(\lambda \star z) = Q(z)$

Generation & Rewriting

Q reduced Gröbner basis of O

$\{g_1, \ldots, g_k\}$ the coefficients of Q

Theorem:

$$\mathbb{K}(z)^G = \mathbb{K}(g_1, \ldots, g_k)$$

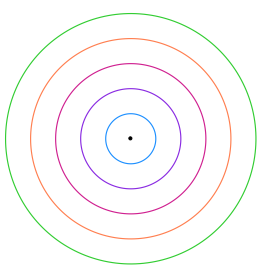
Rewriting $\quad \dfrac{p}{q} \in \mathbb{K}(z)^G$

- $y_1, \ldots, y_k$ a new indeterminates
- $Q_y := Q(g_i \leftarrow y_i)$
- $p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$
- $q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$
- $\dfrac{p(z)}{q(z)} = \dfrac{a_\alpha(g)}{b_\alpha(g)}$

Linear actions in the plane

$$\lambda_1^2 + \lambda_2^2 = 1$$

$$(\lambda_1, \lambda_2) \mapsto \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

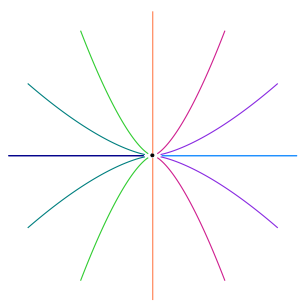


$$Q = \left\{X^2 + Y^2 - (x^2 + y^2)\right\}$$

$$g = x^2 + y^2$$

$$\mathbb{K}^*$$

$$\lambda \mapsto \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^3 \end{pmatrix}$$



$$Q = \left\{Y^2 - \frac{y^2}{x^3} X^3\right\}$$

$$g = \frac{y^2}{x^3}$$

References

H. & Kogan 07, H. 13 this presentation and what follows.

Müller-Quade & Beth 99, Kemper 07:

Case of linear group actions.

Proof: $(Q) = (Z - z) \cap \mathbb{K}(z)^G[Z]$.

Derksen 99 : Polynomial invariants of reductive groups acting linearly

Popov & Vinberg 89

There exists a generating set Q of O the coefficients $\{g_1, \ldots, g_\kappa\}$ of which are in $\mathbb{K}(z)^G$

$\{g_1, \ldots, g_\kappa\}$ separate orbits

A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

Rosenlicht 56

The coefficients of the Chow form of O are rational invariants and separate orbits

A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

Example [Derksen 99]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mapsto \begin{pmatrix} a & b & & & & & \\ c & d & & & & & \\ & & a & b & & & \\ & & c & d & & & \\ & & & & a^2 & ac & c^2 \\ & & & & 2ab & ad+bc & 2cd \\ & & & & b^2 & bd & d^2 \end{pmatrix}$$

$$G = (ad - bc - 1)$$

$$Z - \lambda \star z : \quad Z_1 - (az_1 + bz_2),\ Z_2 - (cz_1 + dz_2),\ Z_3 - (az_3 + bz_4),\ Z_4 - (cz_3 + dz_4)$$
$$Z_5 - (a^2 z_5 + 2abz_6 + b^2 z_7), \quad Z_7 - (c^2 z_5 + 2cdz_6 + d^2 z_7)$$
$$Z_6 - (caz_5 + ad + bcz_6 + bdz_7),$$

$$O = (G + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$$

$$\begin{gathered} {Z_6}^2 - Z_7\, Z_5 + g_1\, g_3 - {g_4}^2, \quad Z_6\, Z_4 + g_3\, g_2\, Z_2 - g_4\, Z_4 - Z_3\, Z_7, \\ Z_5\, Z_4 - Z_3\, Z_6 + g_3\, g_2\, Z_1 - g_4\, Z_3, \quad Z_3\, Z_2 - Z_1\, Z_4 - g_2, \\ Z_2\, Z_6 - Z_1\, Z_7 + g_4\, Z_2 - \frac{g_1}{g_2}\, Z_4, \quad Z_2\, Z_5 + Z_1\, g_4 - Z_6\, Z_1 - \frac{g_1}{g_2}\, Z_3, \\ {Z_2}^2 + \frac{g_1}{g_3\, {g_2}^2}\, {Z_4}^2 - \frac{1}{g_3} Z_7 - 2\, \frac{g_4}{g_3\, g_2}\, Z_4\, Z_2,\ {Z_1}^2 - \frac{1}{g_3} Z_5 - 2\, \frac{g_4}{g_3\, g_2}\, Z_3\, Z_1 + \frac{g_1}{g_3\, {g_2}^2}\, {Z_3}^2 \\ Z_2\, Z_1 - \frac{g_4}{g_3} - \frac{Z_6}{g_3} + \frac{g_1}{g_3\, {g_2}^2}\, Z_4\, Z_3 - 2\, \frac{g_4}{g_3\, g_2}\, Z_4\, Z_1, \end{gathered}$$

2.2 Section to the orbits

Section of degree e

An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

A generic affine space of dimension $n - d$ is a section

where d = dimension of generic orbits

Intersection ideal

$\mathcal{P} = \mathcal{V}(P)$ $\qquad$ P a prime ideal in $\mathbb{K}[Z]$

$$I = (G + (Z - \lambda \star z) + P) \cap \mathbb{K}(z)[Z]$$

I is a zero dimensional ideal and $\dim_{\mathbb{K}} \mathbb{K}(z)[Z]/I = e$

Invariants, rewriting

$$I = (G + (Z - \lambda \star z) + P) \cap \mathbb{K}(z)[Z]$$

Q reduced Gröbner basis of I with coefficients $\{g_1, \ldots, g_k\}$

Thm: $\mathbb{K}(z)^G = \mathbb{K}(g_1, \ldots, g_k)$

Thm: Rewriting for $f \in \mathbb{K}(z)^G$ in terms of $g_1, \ldots, g_k$: *as before*

Prop: If $\mathcal{P}$ is of degree 1 then $I = (Z_1 - g_1(z), \ldots, Z_n - g_n(z))$

Then: $f(z_1, \ldots, z_n) = f(g_1, \ldots, g_n)$. $\qquad$ A simple substitution!

Example [Derksen 99]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mapsto \begin{pmatrix} a & b & & & & \\ c & d & & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & a^2 & ac & c^2 \\ & & & & 2ab & ad+bc & 2cd \\ & & & & b^2 & bd & d^2 \end{pmatrix}$$

$$I = (P + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$$

$$Q = \{Z_1 - 1, Z_2, Z_3, Z_4 - g_2, Z_5 - g_3, Z_6 - g_4, Z_7 - g_1\}$$

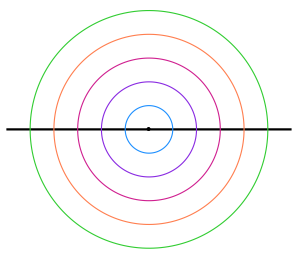
$$g_1 = z_7\, {z_1}^2 - 2\, z_2\, z_6\, z_1 + {z_2}^2 z_5, \quad g_2 = z_3\, z_2 - z_1\, z_4,$$
$$g_3 = \frac{{z_3}^2 z_7 - 2\, z_6\, z_4\, z_3 + z_5\, {z_4}^2}{(z_1\, z_4 - z_3\, z_2)^2},$$
$$g_4 = \frac{z_1\, z_6\, z_4 - z_1\, z_3\, z_7 + z_3\, z_2\, z_6 - z_2\, z_5\, z_4}{z_1\, z_4 - z_3\, z_2}$$

$$f(z) \in \mathbb{K}(z)^G \;\Rightarrow\; f(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = f(1, 0, 0, g_2, g_3, g_4, g_1)$$

Linear actions in the plane

$$\lambda_1^2 + \lambda_2^2 = 1$$

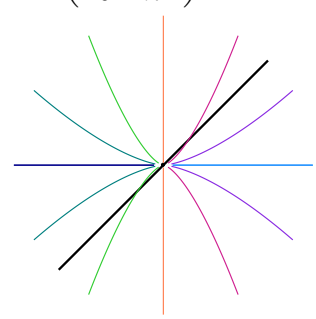
$$(\lambda_1, \lambda_2) \mapsto \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$



$$Q = \{Y, X^2 - (x^2 + y^2)\}$$

$$\mathbb{K}^*$$

$$\lambda \mapsto \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^3 \end{pmatrix}$$



$$Q = \left\{X - \frac{x^3}{y^2}, Y - \frac{x^3}{y^2}\right\}$$

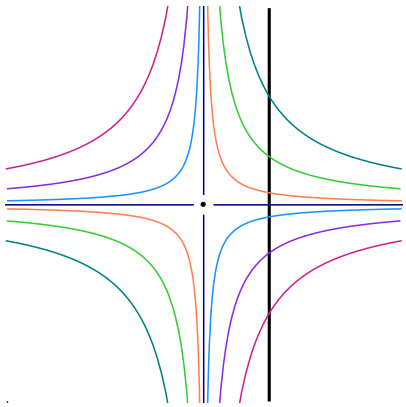
3 Scalings = diagonal representations of tori

Scalings in the plane

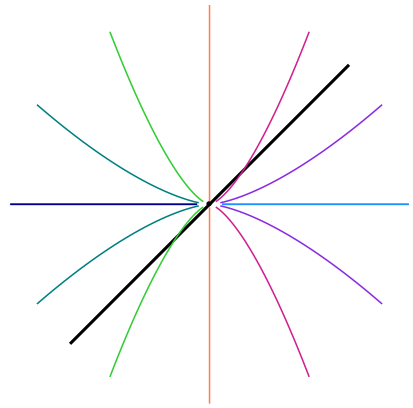
$$\begin{array}{rccc} \rho: & \mathbb{K}^* & \to & \mathrm{GL}(2) \\ & \lambda & \mapsto & \begin{pmatrix} \lambda^a & \cdot \\ \cdot & \lambda^b \end{pmatrix} \end{array}$$

$$\begin{array}{rccc} \star: & \mathbb{K}^* \times \mathbb{K}^2 & \to & \mathbb{K}^2 \\ & (\lambda, (x, y)) & \mapsto & (\lambda^a x, \lambda^b y) \end{array}$$

$$A = \begin{bmatrix} a & b \end{bmatrix}$$



$A = \begin{bmatrix} 1 & -1 \end{bmatrix}$



$A = \begin{bmatrix} 2 & 3 \end{bmatrix}$

Bezout: $\alpha\, a - \beta\, b = 1$

$P = (X^\alpha - Y^\beta) : (XY)^\infty$

$$I = \left(X - \left(\frac{y^a}{x^b}\right)^\beta, Y - \left(\frac{y^a}{x^b}\right)^\alpha \right)$$

Hermite normal form

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \alpha & -b \\ -\beta & a \end{bmatrix}}_{\text{multiplier}} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{Hermite form}} .$$

3.1 Matricial point of view on scalings

Monomial maps

$\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_r \end{bmatrix} \in (\mathbb{K}^*)^r$

$a = \begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \in \mathbb{Z}^r$

$$\lambda^a = \lambda_1^{a_1} \cdots \lambda_r^{a_r}$$

$A = \begin{bmatrix} A_{\cdot,1}, \dots, A_{\cdot,n} \end{bmatrix} \in \mathbb{Z}^{r \times n}$, a matrix of integers

$$\lambda^A = \begin{bmatrix} \lambda^{A_{\cdot,1}} & \dots & \lambda^{A_{\cdot,n}} \end{bmatrix}$$

Monomial maps

$\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_r \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 & \dots & \mu_r \end{bmatrix}$

$$\lambda \star \mu = \begin{bmatrix} \lambda_1\, \mu_1 & \dots & \lambda_r\, \mu_r \end{bmatrix}$$

$(\lambda \star \mu)^A = \lambda^A \star \mu^A$ $\qquad$ $\lambda^{A+B} = \lambda^A \star \lambda^B$ $\qquad$ $\lambda^{AB} = (\lambda^A)^B$

Scalings

$A \in \mathbb{Z}^{r\times n}$ defines the linear action of $(\mathbb{K}^*)^r$

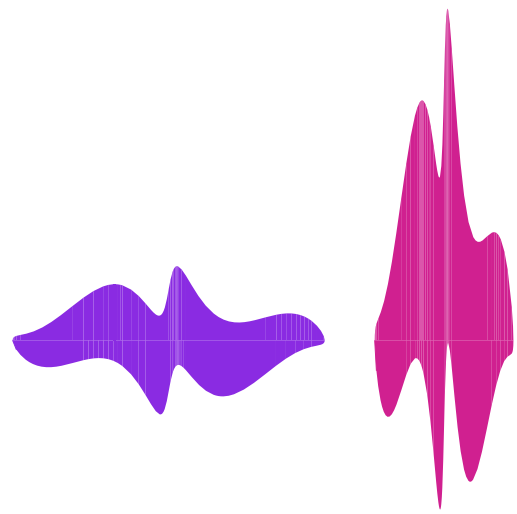
$$\begin{array}{rccc} \rho: & (\mathbb{K}^*)^r & \to & \mathrm{GL}_n(\mathbb{K}) \\ & \lambda & \mapsto & \begin{pmatrix} \lambda^{A\cdot,1} & & \\ & \ddots & \\ & & \lambda^{A\cdot,n} \end{pmatrix} \end{array}$$

$(\mathbb{K}^*)^r \times \mathbb{K}^n$

$(\lambda \ , \ z)$

Ex: $A = \begin{bmatrix} -2 & 3 \end{bmatrix}$ $\qquad \lambda^A \star (x,y) = (\lambda^{-2}\, x, \lambda^3\, y)$

$$\rho : \lambda \mapsto \begin{pmatrix} \lambda^{-2} & \\ & \lambda^3 \end{pmatrix}$$



Rational invariants $\mathbb{K}(z)^A$

Invariant Laurent monomials: $\quad z^v = z_1^{v_1} \dots z_n^{v_n}, \quad v \in \mathbb{Z}^n$

$$(\lambda^A \star z)^v = z^v \ \Leftrightarrow \ A \cdot v = 0$$

Rational Invariants

$$f(z) = \frac{\displaystyle\sum_{v \in \ker A \cap \mathbb{Z}^n} a_v\, z^v}{\displaystyle\sum_{v \in \ker A \cap \mathbb{Z}^n} b_v\, z^v}$$

3.2 Invariants, rewriting and section

Hermite Normal Forms

$H \in \mathbb{Z}^{r\times n}$, rank $r < n$ in (column) Hermite normal form if

$$H = \begin{bmatrix} 7 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \\ & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 2 & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Zero elements in right column.

Upper triangular in left columns with nonnegative entries.

Diagonal entries in left columns largest in each row.

Can always transform any integer matrix A to one in column Hermite form via integer column operations.

i.e.

Hermite Multiplier V

There exists a unimodular $V \in \mathbb{Z}^{n\times n}$ such that $A \cdot V = [H, 0]$

- Unimodular means $W = V^{-1} \in \mathbb{Z}^{n\times n}$
- Useful to write : $A \cdot V = A \cdot [V_{\mathrm{i}}, V_{\mathrm{n}}] = [H, 0]$.
- Columns of V_n form a lattice basis for $\ker A \cap \mathbb{Z}^n$

V not unique.

Generating invariants and rewriting

$$A \in \mathbb{Z}^{r\times n}, \qquad V \in \mathbb{Z}^{n\times n}, \qquad A \cdot V = \begin{bmatrix} H & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} V_{\mathrm{i}} & V_{\mathrm{n}} \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} W_{\mathrm{u}} \\ W_{\mathrm{d}} \end{bmatrix}$$

Thm: The $n - r$ components of $g = [z_1, \ldots, z_n]^{V_{\mathrm{n}}}$ form a generating set of (algebraically independent) rational invariants.

Thm: $$R \in \mathbb{K}(z)^A \quad \Rightarrow \quad R(z) = R\left(g^{W_{\mathrm{d}}}\right).$$

Pf: $I = V_i W_u + V_n W_d$ and $\ker A \subset \ker W_u$:

Example

- Scaling exponent matrix :

$$A = \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix}$$

- Hermite multiplier and its inverse:

$$V = \left[\begin{array}{cc|ccc} 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & & 2 & 1 \\ 0 & 0 & & & 1 \end{array}\right] \quad \text{and} \quad W = \left[\begin{array}{ccccc} 2 & -2 & -2 & 3 & -1 \\ 0 & 3 & 1 & -4 & 3 \\ \hline 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

- Invariants:

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}^{V_n} = \begin{bmatrix} \frac{z_1^2 z_3^3}{z_2} & z_1 z_2^2 z_3^2 z_4^2 & z_3 z_4 z_5 \end{bmatrix}$$

- Rewrite rules:

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix} \to \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}^{W_d} = \begin{bmatrix} \frac{1}{g_2} & \frac{g_2}{g_1} & g_2 & \frac{g_1}{g_2} & \frac{g_3}{g_1} \end{bmatrix}$$

e.g. $\quad z_1 z_2^2 z_3^3 z_4^3 z_5 - \frac{z_1^2 z_3^3}{z_2} - 1 \;\to\; g_2 g_3 - g_1 - 1$

Rational Section

Hyp: $A \in \mathbb{Z}^{r\times n}$, $\qquad V = [\ V_{\mathfrak{i}}\ V_{\mathfrak{n}}\ ] \in \mathbb{Z}^{n\times n}$, $\qquad V^{-1} = \begin{bmatrix} W_{\mathfrak{u}} \\ W_{\mathfrak{d}} \end{bmatrix}$

Thm: $\mathcal{P}:\ z^{V_{\mathfrak{i}}} = 1_r$ is a rational section on $(\mathbb{K}^*)^n$

$$\mathcal{O}_z \cap \mathcal{P} = \left\{ z^{V_{\mathfrak{n}} W_{\mathfrak{d}}} \right\} \text{ for } z \in (\mathbb{K}^*)^n .$$

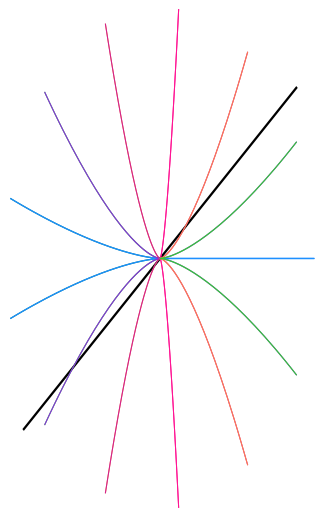
$\diamond\ I = \left(Z - g^{W_{\mathfrak{d}}}\right)$

Example

$$A = [\ 2 \quad 3\ ], \quad V = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}, \quad W = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$g = \frac{z_1^3}{z_2^2}, \quad R(z_1, z_2) = R\,(g, g)$$

$\mathcal{P}:\ \dfrac{z_2}{z_1} = 1$ $\qquad \mathcal{O}_z \cap \mathcal{P} = \left\{ \left(\frac{z_1^3}{z_2^2}, \frac{z_1^3}{z_2^2} \right) \right\}$



3.3 Applications to symmetry reductions

Parameter reduction in biological models

Prey-predator model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} &= \left(\left(1 - \frac{n}{k_1}\right) r - k_2 \frac{p}{n+e} \right) n, \\ \dot{p} &= s\left(1 - h\frac{p}{n}\right) p. \end{cases}$$

r, s, e, h, k_1, k_2 parameters.

$$\begin{cases} \dot{\mathfrak{n}} &= \left(1 - \frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} &= \mathfrak{s}\left(1 - \frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{cases}$$

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters
The new variables

$$\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{e},\ \mathfrak{p} = \frac{h\,p}{e},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

are the invariants of the scaling symmetry of the original system.

Polynomial system solving

$$p_1(z) = 0, \quad \ldots, \quad p_m(z) = 0$$

where $p_1, \ldots, p_m$ in $\mathbb{K}[z_1, \ldots, z_n]$ or in$\mathbb{K}[z, z^{-1}]$

Symmetry condition: $p(z) = 0 \ \Rightarrow\ p(\lambda^A \star z) = 0, \forall \lambda \in (\mathbb{K}^*)^r$.

This is the case if $p_1, \ldots, p_n$ are A-homogeneous,
i.e. for the weights given by the rows of A.

Reduced system $q(y) = p\left(y^{Wd}\right)$ in $\mathbb{K}\left[y_1, \ldots, y_{n-r}, y^{-1}\right]$

Thm: $p(z) = 0,\ z \in (\mathbb{K}^*)^n \quad \Leftrightarrow$

$$\exists \lambda \in (\mathbb{K}^*)^r,\ y \in (\mathbb{K}^*)^{n-r} \text{ s.t. } q(y) = 0 \text{ and } z = \lambda^A \star y^{W_{\mathfrak{d}}}.$$

Rk: In particular binomial ideals are reduced to dimension 0.

4 Linear actions of finite abelian groups

Finite abelian groups can be diagonalised

$\mathcal{G}$ generated by $G_1, \ldots, G_s \in \mathrm{GL}_n(\mathbb{K})$ with $G_i G_j = G_j G_i$ and $G_i^{p_i} = I$

$$G_i = \Xi^{-1} \begin{pmatrix} \xi_i^{a_{i1}} & & \\ & \ddots & \\ & & \xi_i^{a_{in}} \end{pmatrix} \Xi, \quad \xi_i \text{ primitive } p_i\text{-th root of unity}$$

e.g. the pemutation group generated by $\sigma = (n, n-1, \ldots, 1)$

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 \end{bmatrix} = \Xi^{-1} \begin{bmatrix} \xi & & & & \\ & \xi^2 & & & \\ & & \ddots & & \\ & & & \xi^{n-1} & \\ & & & & 1 \end{bmatrix} \Xi, \quad \text{where } \Xi = \left(\xi^{ij}\right)_{i,j}$$

Diagonal representations of $\mathcal{U} = \mathbb{U}_{p_1} \times \ldots \times \mathbb{U}_{p_s}$

$\mathbb{U}_{p_i}$ the group of p_i-th roots of unity.

$A = (a_{ij}) \in \mathbb{Z}^{s \times n}$ defines an n-dimensional representation of $\mathcal{U}$

$$\begin{array}{rcl} \mathbb{U}_{p_1} \times \ldots \times \mathbb{U}_{p_s} & \to & \mathrm{GL}_n(\mathbb{K}) \\ (\lambda_1, \ldots, \lambda_s) & \mapsto & \begin{pmatrix} \lambda_1^{a_{11}} \ldots \lambda_s^{a_{s1}} & & \\ & \ddots & \\ & & \lambda_1^{a_{1n}} \ldots \lambda_s^{a_{sn}} \end{pmatrix} \end{array}.$$

i.e. a linear action

$$\begin{array}{rcl} \mathcal{U} \times \mathbb{K}^n & \to & \mathbb{K}^n \\ (\lambda, z) & \mapsto & \lambda^A \star z \end{array}.$$

Invariants for $\mathcal{U} = \mathbb{U}_{p_1} \times \ldots \times \mathbb{U}_{p_s}$

Exponent matrix $A = (a_{ij}) \in \mathbb{Z}^{s \times n}$

For $v \in \mathbb{Z}^n$

$$z^v \text{ invariant} \quad \Leftrightarrow \quad (\lambda^A \star z)^v = z^v \quad \Leftrightarrow \quad A\,v \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \mod \begin{bmatrix} p_1 \\ \vdots \\ p_s \end{bmatrix}$$

Order matrix $P = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_s \end{pmatrix}$

$$z^v \text{ invariant} \quad \Leftrightarrow \quad \exists u \in \mathbb{Z}^s \text{ s.t. } A\,v = Pu$$

Invariants from Hermite multiplier

Exponents $A \in \mathbb{Z}^{s \times n}$ Orders $P = \mathrm{diag}(p_1, \ldots, p_s)$

$$\begin{bmatrix} A & -P \end{bmatrix} \begin{bmatrix} V_{\mathfrak{i}} & V_{\mathfrak{n}} \\ P_{\mathfrak{i}} & P_{\mathfrak{n}} \end{bmatrix} = \begin{bmatrix} H & 0 \end{bmatrix}$$

Thm: The n components of $g = z^{V_{\mathfrak{n}}}$ form an algebraically independent generating set of rational invariants.

$$V^{-1} = \begin{bmatrix} W_{\mathfrak{u}} & P_{\mathfrak{u}} \\ W_{\mathfrak{d}} & P_{\mathfrak{d}} \end{bmatrix}$$

Thm: If $f \in \mathbb{K}(z)^{\mathcal{U}}$ then

$$f(z) = f\left(g^{W_{\mathfrak{d}} - P_{\mathfrak{d}} P_{\mathfrak{u}}{}^{-1} W_{\mathfrak{u}}}\right)$$

$A = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}$ $\qquad P = \begin{bmatrix} p \end{bmatrix}$

We are thus considering the group generated by

$$\xi I_n = \begin{bmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{bmatrix}, \quad \xi \text{ a primitive } p\text{th root of unity.}$$

Hermite multipliers

$$V = \begin{bmatrix} 1 & p & p-1 & \dots & p-1 \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & 1 & 1 & \dots & 1 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 1 & 1 & \dots & 1 & -p \\ 0 & -1 & \dots & -1 & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}.$$

Generating invariants

$$g_1 = z_1^p, \text{ and } g_k = z_1^{p-1} z_k, \ 2 \leq k \leq n,$$

Rewrite rules:

$$z_1 \to g_1^{\frac{1}{p}}, \text{ and } z_k \to \frac{g_k}{{g_1}^{\frac{p-1}{p}}}, \ 2 \leq k \leq n.$$

For $u \in \mathbb{N}^n$ s.t. $u_1 + \ldots + u_n = p$ $\qquad z^u \in \mathbb{K}(z)^{\mathcal{U}}$

$$z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n} = g_1^{\frac{u_1}{p} - \frac{u_2(p-1)}{p} - \ldots - \frac{u_n(p-1)}{p}} g_2^{u_1} \ldots g_n^{u_n} = \frac{g_1 g_2^{u_1} \ldots g_n^{u_n}}{g_1^{u_2 + \cdots + u_n}}.$$

Polynomial generators

We can arrange

$$V_n = \begin{pmatrix} m_1 & v_{12} & \dots & v_{1n} \\ & m_2 & \dots & v_{2n} \\ & & \ddots & \\ & & & m_n \end{pmatrix}$$

with $0 \leq v_{ij} < m_i$ and $m_1 \ldots m_n = \frac{p_1 \ldots p_s}{\det H}$

Thm If $\mathcal{G}$ is finite abelian group of $n \times n$ matrices then $\mathbb{K}(z_1, \ldots, z_n)^{\mathcal{G}}$ is generated by n polynomial invariants.

(provided $\mathbb{K}$ contains the p-th roots of unity)

Rk The number of generators for $\mathbb{K}[z_1, \ldots, z_n]^{\mathcal{G}}$ can be combinatorial.

Symmetry reduction schemes

- Systems of polynomial equations

  e.g. homogeneous for a $\mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_s}$ grading

- Dynamical systems $\dot{x} = F(x)$

  e.g. for symmetry under a cyclic group of permutations.

References

E. Hubert, Rational Invariants of a Group Action. Les cours du CIRM 3:1 http://ccirm.cedram.org/item?id=CCIRM_2013__3_1_A3_0 (2013)

E. Hubert, I. Kogan, Rational Invariants of a Group Action. Construction and Rewriting. J. of Symbolic Computation, 42:1-2 (2007).

E. Hubert, I. Kogan, Smooth and Algebraic Invariants of a Group Action. Local and Global Constructions. Foundations Computational Mathematics, 7:4 (2007).

E. Hubert, G. Labahn, Rational invariants of scalings from Hermite normal forms. ISSAC 2012 .

E. Hubert, G. Labahn, Scaling invariants and symmetry reduction of dynamical systems. Foundations of Computational Mathematics 13:4 (2013).

E. Hubert, G. Labahn, Rational Invariants of Finite Abelian Groups. http://hal.inria.fr/hal-00921905