Pseudo H-type algebras and existence of an integer structure.

Irina Markina

University of Bergen, Norway

joint work with Mauricio Godoy, Kenro Furutani, and Alexander Vasiliev

Heisenberg algebra

$\mathcal{N} = \mathrm{span}\{X, Y, Z\} = \mathrm{span}\{X, Y\} \oplus \mathrm{span}\{Z\} = V \oplus \mathcal{Z}$,

$[X, Y] = Z$ is unique non vanishing commutator

Let $(\cdot\,,\cdot)$ be an inner product such that X, Y, Z are orthonormal. Define

$\mathrm{ad}_v(\cdot) := [v, \cdot] \colon V \to \mathcal{Z}$ and $G = (\ker(\mathrm{ad}_v))^{\perp}$. Then

$\mathrm{ad}_v(\cdot) \colon G \to \mathcal{Z}$ is an isometry $\forall\, v \in V$, $(v, v) = 1$

Heisenberg type algebras

Let $(\mathcal{N} = V \oplus_\perp \mathcal{Z}, [\cdot\,,\cdot], (\cdot\,,\cdot))$ be a two step nilpotent Lie algebra. Define

$$\mathrm{ad}_v(\cdot) := [v, \cdot] \colon V \to \mathcal{Z}, \quad G = (\ker(\mathrm{ad}_v))^\perp$$

Require that

$$\mathrm{ad}_v \colon \big(G, (\cdot\,,\cdot)_G\big) \leftrightarrow \big(\mathcal{Z}, (\cdot\,,\cdot)_\mathcal{Z}\big)$$

is an isometry for all $v \in V$ with $(v,v)_V = 1$.
Then $\mathcal{N}$ is an H-type algebra.

Heisenberg type algebra

Define $(J(z,v),u)_V := (z, \operatorname{ad}_v u)_{\mathcal{Z}} = (z,[v,u])_{\mathcal{Z}}$.

- $J\colon \mathcal{Z}\times V \to V$ is a bilinear map
- J is skew symmetric with respect to $(\cdot\,,\cdot)_V$:

$$(J(z,v),u)_V = -(v, J(z,u))_V .$$

- $J(\cdot, v) = \operatorname{ad}_v^*(\cdot)$,
- $J(\cdot, v) = \operatorname{ad}_v^{-1}(\cdot)$, since

$$(J(\operatorname{ad}_v u, v), u')_V = (\operatorname{ad}_v u, \operatorname{ad}_v u')_{\mathcal{Z}} = (u,u')_V$$

Heisenberg type algebra

Define $(J(z,v),u)_V := (z, \mathrm{ad}_v\, u)_{\mathcal{Z}}$. Then

- $J(\mathrm{ad}_v(\cdot), v) = \mathrm{Id}_V$ if we understand

$$\mathrm{ad}_v \colon \big(G, (\cdot\,,\cdot)_G\big) \leftrightarrow (\mathcal{Z}, (\cdot\,,\cdot)_{\mathcal{Z}})$$

$$J(\cdot, v) \colon (\mathcal{Z}, (\cdot\,,\cdot)_{\mathcal{Z}}) \leftrightarrow \big(G, (\cdot\,,\cdot)_G\big)$$

- $J(\cdot, v) \colon \mathcal{Z} \to V$ is an isometry for $(v,v)_V = 1$

$$(J(z,v), J(z,v))_V = (z,z)_{\mathcal{Z}}(v,v)_V$$

Composition of quadr. forms

A bilinear map $\mu\colon U \times V \to V$ is a composition of $(U, \langle\cdot\,,\cdot\rangle_U)$ and $(V, \langle\cdot\,,\cdot\rangle_V)$ if

$$\langle \mu(u,v), \mu(u,v)\rangle_V = \langle u,u\rangle_U \langle v,v\rangle_V$$

for any $u \in U$, $v \in V$. Map μ is normalized if

$$\exists \quad u_0 \in U, \text{ such that } \langle u_0, u_0\rangle_U = 1, \quad \mu(u_0, v) = v$$

Let $\mathcal{Z} = (\text{span}\{u_0\})^{\perp}$, then $J = \mu|_{\mathcal{Z}\times V}$

$$\langle J(z,v), v\rangle_V = \langle \mu(z,v), \mu(u_0,v)\rangle_V = \underbrace{\langle z, u_0\rangle_U}_{=0} \langle v,v\rangle_V = 0$$

Composition of quadr. forms

$J\colon \mathcal{Z} \times V \to V$ with $\langle J(z,v), v\rangle_V = 0$

$$\Longrightarrow \quad \langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V$$

Then we can define a Lie structure on $\mathcal{Z} \times V$ by

$$\langle J(z,v), v'\rangle_V =: \langle z, [v,v']\rangle_{\mathcal{Z}}$$

and $\mathcal{Z}$ becomes a center such that

$$\mathcal{Z} \oplus_{\perp} V \quad \text{w.r.to} \quad \langle \cdot\,,\cdot\rangle = \langle \cdot\,,\cdot\rangle_{\mathcal{Z}} + \langle \cdot\,,\cdot\rangle_V$$

Heisenberg type algebra

Theorem, A. Kaplan, 1981

All H-type algebras arises from compositions of positive definite quadratic forms

$$(\mu(u,v),\mu(u,v))_V = (u,u)_U(v,v)_V.$$

If μ is normalised, then $J = \mu|_{\mathcal{Z}\times V}$ is skew sym.

$$(J(z,v),v')_V =: (z,[v,v'])_{\mathcal{Z}} \quad \Longrightarrow$$

$\Big(\mathcal{Z}\oplus_\perp V, [\cdot\,,\cdot], (\cdot\,,\cdot) = (\cdot\,,\cdot)_{\mathcal{Z}} + (\cdot\,,\cdot)_V\Big)$ is H-type alg.

General H-type algebras

Let $(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$ be a two step nilpotent Lie algebra such that

$$(\mathcal{Z}, \langle\cdot\,,\cdot\rangle_{\mathcal{Z}}), \quad (V, \langle\cdot\,,\cdot\rangle_V) \text{ are non degenerate}$$

Define $\mathrm{ad}_v(\cdot) := [v, \cdot]\colon V \to \mathcal{Z}$. Require that

$$\mathrm{ad}_v\colon \Big(G, \langle\cdot\,,\cdot\rangle_G\Big) \leftrightarrow \Big(\mathcal{Z}, \langle\cdot\,,\cdot\rangle_{\mathcal{Z}}\Big)$$

is (anti-)isometry for all $v \in V$ with $\langle v, v\rangle_V = \pm 1$.
Then $\mathcal{N}$ is a general H-type alg.

General H-type algebras

Theorem, M. Godoy, A. Korolko, I. Markina, 2012

All general H-type algebras arises from decompositions of non-degenerate indefinite quadratic forms

Relation to Clifford algebras

$$\langle J(z,v), J(z,v')\rangle_V = \langle z,z\rangle_{\mathcal{Z}}\langle v,v'\rangle_V. \quad (1)$$

$$\langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V \quad (2)$$

$$\langle J^2(z,v), v'\rangle_V = -\langle J(z,v), J(z,v')\rangle_V = -\langle z,z\rangle_{\mathcal{Z}}\langle v,v'\rangle_V$$

$$\langle (-\langle z,z\rangle_{\mathcal{Z}})v, v'\rangle_V \quad \Longrightarrow$$

$J^2(z,v) = -\langle z,z\rangle_{\mathcal{Z}} v$ or $J^2(z,\cdot) = -\langle z,z\rangle_{\mathcal{Z}}\,\mathrm{Id}_V$ (3)

$$(1)+(2) \quad \Longrightarrow \quad (3)$$

Clifford algebra

$$(1) \qquad \langle J(z,v), J(z,v')\rangle_V = \langle z,z\rangle_{\mathcal{Z}}\langle v,v'\rangle_V.$$

$$(2) \qquad \langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V$$

$$(3) \qquad J^2(z,\cdot) = -\langle z,z\rangle_{\mathcal{Z}}\,\mathrm{Id}_V$$

The last property is defining property for Clifford algebra.

Clifford algebra

Let $(W, \langle \cdot\,,\cdot \rangle_W)$ be a scalar product space.

The Clifford algebra $\mathrm{Cl}((W, \langle \cdot\,,\cdot \rangle_W))$ is an associative algebra with unit $\mathbb{I}$, product $\otimes$, factorized by the relation

$$w \otimes w = -\langle w, w \rangle_W \mathbb{I} \quad \text{or} \quad \Big(w \otimes u + u \otimes w = -2 \langle w, u \rangle_W \mathbb{I} \Big)$$

Clifford algebra

Let $(W, \langle \cdot\,, \cdot \rangle_W)$ be a scalar product space.

The Clifford algebra $\mathrm{Cl}((W, \langle \cdot\,, \cdot \rangle_W))$ is an associative algebra with unit $\mathbb{I}$, product $\otimes$, factorized by the relation

$$w \otimes w = -\langle w, w \rangle_W \mathbb{I} \quad \text{or} \quad \Big(w \otimes u + u \otimes w = -2\langle w, u \rangle_W \mathbb{I} \Big)$$

If $(w_1, \ldots, w_n)$ is an orthonormal basis of W

$$w_k \otimes w_k = -\langle w_k, w_k \rangle_W \mathbb{I}, \quad w_k \otimes w_l = -w_l \otimes w_k, \quad k \neq l$$

Clifford module

The algebra homomorphism J:

$$J\colon\ \mathrm{Cl}(W,\langle\cdot\,,\cdot\rangle_W) \to \mathrm{End}(V)$$

is called representation and (V,J) is

Clifford module for $\mathrm{Cl}(W,\langle\cdot\,,\cdot\rangle_W)$

$$w \mapsto J(w,\cdot)\colon V \to V$$

$$\begin{array}{rcl} w\otimes w & \mapsto & J\circ J = J^2(w,\cdot)\colon V\to V \\ -\langle w,w\rangle_W \mathbb{I} & \mapsto & -\langle w,w\rangle_W \,\mathrm{Id}_V \end{array}$$

Relation to Clifford module

$$(1) \qquad \langle J(z,v), J(z,v')\rangle_V = \langle z,z\rangle_{\mathcal{Z}}\langle v,v'\rangle_V.$$

$$(2) \qquad \langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V$$

$$(3) \qquad J^2(z,\cdot) = -\langle z,z\rangle_{\mathcal{Z}}\,\mathrm{Id}_V$$

Question: given (3) can we construct a general H-type algebra?

$$(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$$

Relation to Clifford module

$$(1) \qquad \langle J(z,v), J(z,v')\rangle_V = \langle z,z\rangle_{\mathcal{Z}} \langle v,v'\rangle_V.$$

$$(2) \qquad \langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V$$

$$(3) \qquad J^2(z,\cdot) = -\langle z,z\rangle_{\mathcal{Z}}\, \mathrm{Id}_V$$

Question: given (3) can we construct a general H-type algebra?

$$(\mathcal{N} = V \oplus_\perp \mathcal{Z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$$

Answer: yes if we add (1) or (2)

Relation to Clifford module

If for Clifford $\mathrm{Cl}(\mathcal{Z}, \langle \cdot\,, \cdot \rangle_{\mathcal{Z}})$-module V

$$\left((3) \qquad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \, \mathrm{Id}_V \right)$$

there is a scalar product $\langle \cdot\,, \cdot \rangle_V$ such that

$$(2) \qquad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

hold, then $(V, \langle \cdot\,, \cdot \rangle_V)$ is called admissible.

Relation to Clifford module

If for Clifford $\mathrm{Cl}(\mathcal{Z}, \langle \cdot \,, \cdot \rangle_{\mathcal{Z}})$-module V

$$\left((3) \qquad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \, \mathrm{Id}_V \right)$$

there is a scalar product $\langle \cdot \,, \cdot \rangle_V$ such that

$$(2) \qquad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

hold, then $(V, \langle \cdot \,, \cdot \rangle_V)$ is called admissible.

Do admissible modules exist?

It is known

Given normalised composition

$$\langle \mu(u,v), \mu(u,v)\rangle_V = \langle u,u\rangle_U \langle v,v\rangle_V$$

the skew symmetric map $J(z,v) = \mu|_{\mathcal{Z}\times V}$ defines the representation of algebra $\mathrm{Cl}(\mathcal{Z}, \langle\cdot\,,\cdot\rangle_{\mathcal{Z}})$ on V.

Given an admissible $\mathrm{Cl}(\mathcal{Z}, \langle\cdot\,,\cdot\rangle_{\mathcal{Z}}) -$ module V :

$$(2) \quad \langle J(z,v), v'\rangle_V = -\langle v, J(z,v')\rangle_V, \quad \text{for all } z \in \mathcal{Z}$$

there is $U = \mathcal{Z} \oplus U_0$ such that composition holds.

Existence of adm. module

Given a $\mathrm{Cl}(\mathcal{Z}, \langle \cdot\,, \cdot \rangle_{\mathcal{Z}})$-module V, then V or $V \oplus V$ can be equipped with a scalar product satisfying

$$(2) \quad \langle J(z,v), v' \rangle_V = -\langle v, J(z,v') \rangle_V, \quad \text{for all } z \in \mathcal{Z}$$

P. Ciatti, 2000. Moreover

$$(V, \langle \cdot\,, \cdot \rangle_V) \quad \text{or} \quad (V \oplus V, \langle \cdot\,, \cdot \rangle_{V \oplus V})$$

is a neutral space

Classical H-type algebras

For $(\mathcal{N} = V \oplus_\perp \mathcal{Z}, [\cdot\,,\cdot], (\cdot\,,\cdot))$ there is a basis $V = \mathrm{span}\{v_\alpha\}$ and $\mathcal{Z} = \mathrm{span}\{z_j\}$ such that

$$[v_\alpha, v_\beta] = \sum_j C^j_{\alpha\beta} z_j, \qquad C^j_{\alpha\beta} \in \mathbb{Z}.$$

G. Crandall, J. Dodziuk, *Integral structures on H-type Lie algebras*, J. Lie Theory **12** (2002), no. 1, 69-79.
P. Eberlein, *Geometry of 2-step nilpotent Lie groups, Modern dynamical systems and applications*, Cambridge Univ. Press, Cambridge (2004), 67–101.

A. I. Malćev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation 39, 1951; Izv.

Akad. Nauk USSR, Ser. Mat. **13** (1949), 9-32.

General H-type algebras

Do the general H-type algebras

$$(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$$

admit integer constants?

$$[v_\alpha, v_\beta] = \sum_j C^j_{\alpha\beta} z_j, \qquad C^j_{\alpha\beta} \in \mathbb{Z}.$$

Answer is YES!

Isomorphisms preserving admissibility

$$\mathrm{Cl}_{r+8,s} \sim \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{8,0} \sim \mathrm{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\mathrm{Cl}_{r,s+8} \sim \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{0,8} \sim \mathrm{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\mathrm{Cl}_{r+4,s+4} \sim \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{4,4} \sim \mathrm{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\mathrm{Cl}_{r,s+1} \sim \mathrm{Cl}_{s,r+1}$$

Main idea of the proof

$$[v_\alpha, v_\beta] = \sum_j C^j_{\alpha\beta} z_j, \quad v_\alpha \in V,\ z \in \mathcal{Z}.$$

$$J\colon \mathcal{Z} \to \mathrm{End}(V)$$

$$J_{z_j} v_\alpha = \sum_\beta B^j_{\alpha\beta} v_\beta.$$

$$\langle J_{z_j} v_\alpha, v_\beta \rangle_V = \langle [v_\alpha, v_\beta], z_j \rangle_{\mathcal{Z}}$$

$$C^j_{\alpha\beta} = B^j_{\alpha\beta} \nu^{\mathcal{Z}}_j \nu^V_\beta$$

Main idea of the proof

$v_1 = w,\ v_2 = J_{z_1} w,\ \dots\ ,\ v_{n+1} = J_{z_n} w, \quad n = \dim \mathcal{Z}$

is an orthogonal basis and since

$$\langle J_{z_j} v_\alpha, v_\beta \rangle_V = 0, \pm 1$$

we need to include $J_{z_i} J_{z_j} w$. But they can be non-orthogonal to $v_1, \dots, v_{n+1}$.

If $m = \dim V$, then $m = 2^{\mu + r}$, where $\mu = 0, 1, 2$ and $2r = n, n-1, n-2, n-3$, $r \in \mathbb{N}$.

Main idea of the proof

Find operators $P_k : V \to V$ such that

1. $P_k^2 = \mathrm{Id}_V \quad \longrightarrow \quad P_k w = w$
2. $\langle P_k v, P_k v\rangle_V = \langle v, v\rangle_V$
3. $P_k P_l = P_l P_k$

Typical operator is $P = J_{z_i} J_{z_j} J_{z_l} J_{z_m}$. We choose $w \in V$

$$P_k w = w \quad \text{for all} \quad k$$

Main idea of the proof

We need such operators P_k to be sure that the following holds

a. $\langle w, w \rangle_V$=1

b. $\{w,\ J_{z_i}w,\ J_{z_i}J_{z_j}w,\ J_{z_i}J_{z_j}J_{z_l}w,\ \ J_{z_i}J_{z_j}J_{z_l}J_{z_m}w\}$, $1 \leq i < j < l < m \leq \dim V$ is an o.n. basis

c. J_{z_i} permute the basis for all $i = 1, \ldots, \dim \mathcal{Z}$

Example $\mathrm{Cl}_{0,6} \sim \mathbb{H}(4)$

Involution\Generator	J_1	J_2	J_3	J_4	J_5	J_6,
$P_1 = J_1J_2J_3J_4$	a	a	a	a	c	c
$P_2 = J_1J_2J_5J_6$	a	a	c	c	a	a

Inv.	Eigenvalues			
P_1	$+1$		-1	
P_2	$+1$	-1	$+1$	-1
Eigenv.	$w,\ J_1J_2w$ $J_1J_3J_5w,\ J_1J_3J_6w$	$J_1J_3w,\ J_1J_4w$ $J_5w,\ J_6w$	$J_1J_5w,\ J_1J_6w$ $J_3w,\ J_4w$	$J_3J_5w,\ J_3J_6w$ $J_1w,\ J_2w$

Example $\mathrm{Cl}_{0,6} \sim \mathbb{H}(4)$

Inv.\Comp. op.	$J_1(+ \to -)$	$J_5(+ \to -)$	$J_2J_3J_5(+ \to -)$
P_1	a	c	c
P_2		a	c

Inv.	Eigenvalues			
P_1	$+1$		-1	
P_2	$+1$	-1	$+1$	-1
Eigenv.	$w,\ J_1J_2w$ $J_1J_3J_5w,\ J_1J_3J_6w$	$J_1J_3w,\ J_1J_4w$ $J_5w,\ J_6w$	$J_1J_5w,\ J_1J_6w$ $J_3w,\ J_4w$	$J_3J_5w,\ J_3J_6w$ $J_1w,\ J_2w$

Example $\mathrm{Cl}_{0,6} \sim \mathbb{H}(4)$

$$J_1 w = J_2 J_3 J_4 w = J_2 J_5 J_6 w,$$
$$J_2 w = -J_1 J_3 J_4 w = -J_1 J_5 J_6 w,$$
$$J_3 w = J_1 J_2 J_4 w = -J_4 J_5 J_6 w,$$
$$J_4 w = -J_1 J_2 J_3 w = J_3 J_5 J_6 w,$$
$$J_5 w = J_1 J_2 J_6 w = -J_3 J_4 J_6 w,$$
$$J_6 w = -J_1 J_2 J_5 w = J_3 J_4 J_5 w,$$

$$J_1 J_3 J_5 w = -J_1 J_4 J_6 w = J_2 J_3 J_6 w = J_2 J_4 J_5 w,$$
$$J_1 J_3 J_6 w = J_1 J_4 J_5 w = -J_2 J_3 J_5 w = J_2 J_4 J_6 w.$$

$$J_1 J_2 w = -J_3 J_4 w = -J_5 J_6 w,$$
$$J_1 J_3 w = J_2 J_4 w,$$
$$J_1 J_4 w = -J_2 J_3 w,$$
$$J_1 J_5 w = J_2 J_6 w,$$
$$J_1 J_6 w = -J_2 J_5 w,$$
$$J_3 J_5 w = -J_4 J_6 w,$$
$$J_3 J_6 w = J_4 J_5 w.$$

The end