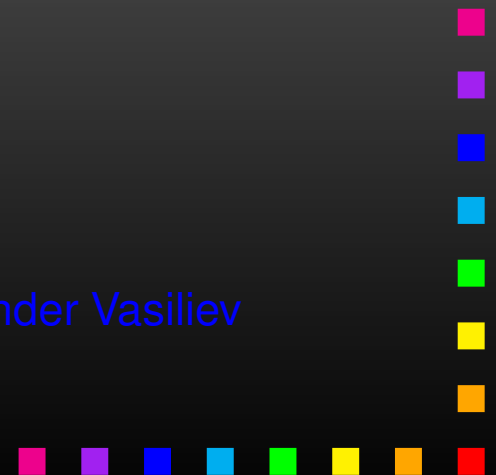


Pseudo H -type algebras and existence of an integer structure.

Irina Markina

University of Bergen, Norway

joint work with Mauricio Godoy, Kenro Furutani, and Alexander Vasiliev



Heisenberg algebra

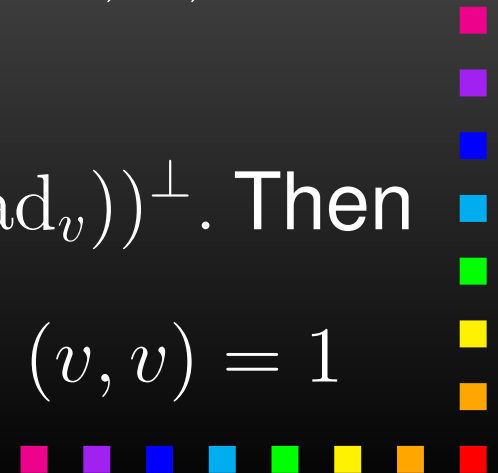
$$\mathcal{N} = \text{span}\{X, Y, Z\} = \text{span}\{X, Y\} \oplus \text{span}\{Z\} = V \oplus \mathcal{Z},$$

$[X, Y] = Z$ is unique non vanishing commutator

Let (\cdot, \cdot) be an inner product such that X, Y, Z are orthonormal. Define

$\text{ad}_v(\cdot) := [v, \cdot]: V \rightarrow \mathcal{Z}$ and $G = (\ker(\text{ad}_v))^\perp$. Then

$\text{ad}_v(\cdot): G \rightarrow \mathcal{Z}$ is an isometry $\forall v \in V, (v, v) = 1$



Heisenberg type algebras

Let $(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], (\cdot, \cdot))$ be a two step nilpotent Lie algebra. Define

$$\text{ad}_v(\cdot) := [v, \cdot]: V \rightarrow \mathcal{Z}, \quad G = (\ker(\text{ad}_v))^{\perp}$$

Require that

$$\text{ad}_v: (G, (\cdot, \cdot)_G) \leftrightarrow (\mathcal{Z}, (\cdot, \cdot)_{\mathcal{Z}})$$

is an isometry for all $v \in V$ with $(v, v)_V = 1$.

Then \mathcal{N} is an H-type algebra.



Heisenberg type algebra

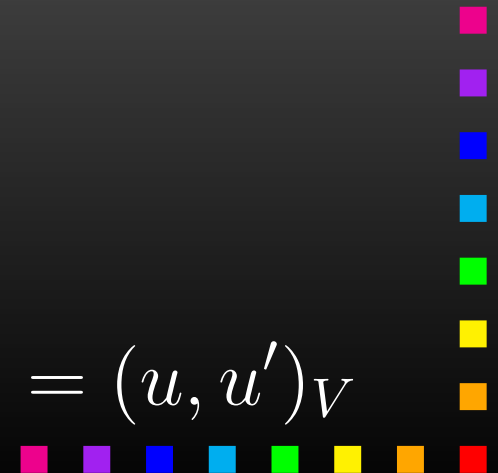
Define $(J(z, v), u)_V := (z, \text{ad}_v u)_Z = (z, [v, u])_Z$.

- $J: Z \times V \rightarrow V$ is a bilinear map
- J is skew symmetric with respect to $(\cdot, \cdot)_V$:

$$(J(z, v), u)_V = -(v, J(z, u))_V.$$

- $J(\cdot, v) = \text{ad}_v^*(\cdot)$,
- $J(\cdot, v) = \text{ad}_v^{-1}(\cdot)$, since

$$(J(\text{ad}_v u, v), u')_V = (\text{ad}_v u, \text{ad}_v u')_Z = (u, u')_V$$



Heisenberg type algebra

Define $(J(z, v), u)_V := (z, \text{ad}_v u)_Z$. Then

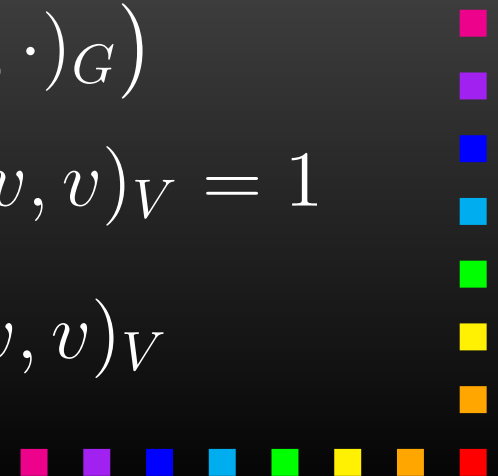
- $J(\text{ad}_v(\cdot), v) = \text{Id}_V$ if we understand

$$\text{ad}_v: (G, (\cdot, \cdot)_G) \leftrightarrow (\mathcal{Z}, (\cdot, \cdot)_Z)$$

$$J(\cdot, v): (\mathcal{Z}, (\cdot, \cdot)_Z) \leftrightarrow (G, (\cdot, \cdot)_G)$$

- $J(\cdot, v): \mathcal{Z} \rightarrow V$ is an isometry for $(v, v)_V = 1$

$$(J(z, v), J(z, v))_V = (z, z)_Z (v, v)_V$$



Composition of quadr. forms

A bilinear map $\mu: U \times V \rightarrow V$ is a **composition** of $(U, \langle \cdot, \cdot \rangle_U)$ and $(V, \langle \cdot, \cdot \rangle_V)$ if

$$\langle \mu(u, v), \mu(u, v) \rangle_V = \langle u, u \rangle_U \langle v, v \rangle_V$$

for any $u \in U, v \in V$. Map μ is **normalized** if

$$\exists u_0 \in U, \text{ such that } \langle u_0, u_0 \rangle_U = 1, \quad \mu(u_0, v) = v$$

Let $\mathcal{Z} = (\text{span}\{u_0\})^\perp$, then $J = \mu|_{\mathcal{Z} \times V}$

$$\langle J(z, v), v \rangle_V = \langle \mu(z, v), \mu(u_0, v) \rangle_V = \underbrace{\langle z, u_0 \rangle_U}_{=0} \langle v, v \rangle_V = 0$$



Composition of quadr. forms

$$J: \mathcal{Z} \times V \rightarrow V \text{ with } \langle J(z, v), v \rangle_V = 0$$

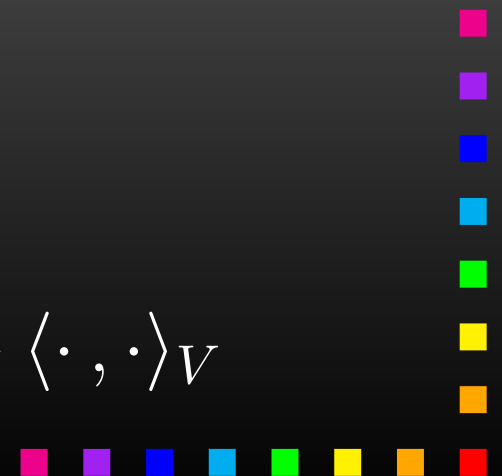
$$\implies \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

Then we can define a Lie structure on $\mathcal{Z} \times V$ by

$$\langle J(z, v), v' \rangle_V =: \langle z, [v, v'] \rangle_{\mathcal{Z}}$$

and \mathcal{Z} becomes a center such that

$$\mathcal{Z} \oplus_{\perp} V \text{ w.r.to } \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{Z}} + \langle \cdot, \cdot \rangle_V$$



Heisenberg type algebra

Theorem, A. Kaplan, 1981

All H -type algebras arises from compositions of positive definite quadratic forms

$$(\mu(u, v), \mu(u, v))_V = (u, u)_U (v, v)_V.$$

If μ is normalised, then $J = \mu|_{\mathcal{Z} \times V}$ is skew sym.

$$(J(z, v), v')_V =: (z, [v, v'])_{\mathcal{Z}} \implies$$

$(\mathcal{Z} \oplus_{\perp} V, [\cdot, \cdot], (\cdot, \cdot) = (\cdot, \cdot)_{\mathcal{Z}} + (\cdot, \cdot)_V)$ is H -type alg.



General H -type algebras

Let $(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ be a two step nilpotent Lie algebra such that

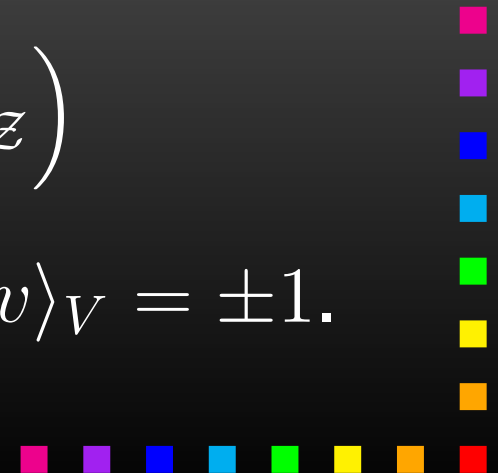
$(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$, $(V, \langle \cdot, \cdot \rangle_V)$ are non degenerate

Define $\text{ad}_v(\cdot) := [v, \cdot]: V \rightarrow \mathcal{Z}$. Require that

$$\text{ad}_v: \left(G, \langle \cdot, \cdot \rangle_G \right) \leftrightarrow \left(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}} \right)$$

is (anti-)isometry for all $v \in V$ with $\langle v, v \rangle_V = \pm 1$.

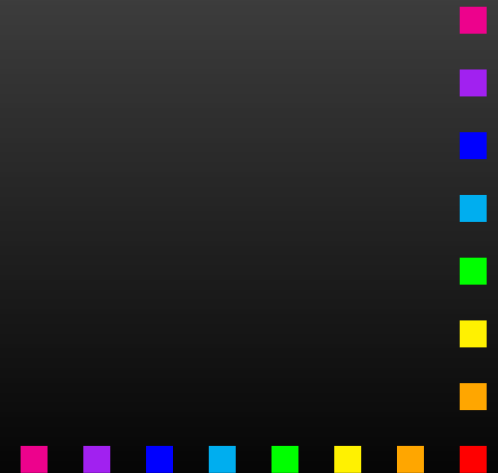
Then \mathcal{N} is a general H -type alg.



General H -type algebras

Theorem, M. Godoy, A. Korolko, I. Markina, 2012

All general H -type algebras arises from decompositions of non-degenerate indefinite quadratic forms



Relation to Clifford algebras

$$\langle J(z, v), J(z, v') \rangle_V = \langle z, z \rangle_{\mathcal{Z}} \langle v, v' \rangle_V. \quad (1)$$

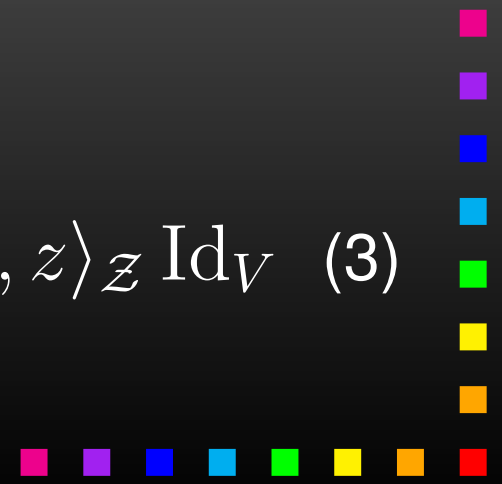
$$\langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V \quad (2)$$

$$\langle J^2(z, v), v' \rangle_V = -\langle J(z, v), J(z, v') \rangle_V = -\langle z, z \rangle_{\mathcal{Z}} \langle v, v' \rangle_V$$

$$\langle (-\langle z, z \rangle_{\mathcal{Z}})v, v' \rangle_V \implies$$

$$J^2(z, v) = -\langle z, z \rangle_{\mathcal{Z}} v \text{ or } J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V \quad (3)$$

$$(1) + (2) \implies (3)$$



Clifford algebra

$$(1) \quad \langle J(z, v), J(z, v') \rangle_V = \langle z, z \rangle_{\mathcal{Z}} \langle v, v' \rangle_V.$$

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

$$(3) \quad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V$$

The last property is defining property for Clifford algebra.



Clifford algebra

Let $(W, \langle \cdot, \cdot \rangle_W)$ be a scalar product space.

The Clifford algebra $Cl((W, \langle \cdot, \cdot \rangle_W))$ is an associative algebra with unit \mathbb{I} , product \otimes , factorized by the relation

$$w \otimes w = -\langle w, w \rangle_W \mathbb{I} \quad \text{or} \quad \left(w \otimes u + u \otimes w = -2\langle w, u \rangle_W \mathbb{I} \right)$$



Clifford algebra

Let $(W, \langle \cdot, \cdot \rangle_W)$ be a scalar product space.

The Clifford algebra $\text{Cl}((W, \langle \cdot, \cdot \rangle_W))$ is an associative algebra with unit \mathbb{I} , product \otimes , factorized by the relation

$$w \otimes w = -\langle w, w \rangle_W \mathbb{I} \quad \text{or} \quad (w \otimes u + u \otimes w = -2\langle w, u \rangle_W \mathbb{I})$$

If (w_1, \dots, w_n) is an orthonormal basis of W

$$w_k \otimes w_k = -\langle w_k, w_k \rangle_W \mathbb{I}, \quad w_k \otimes w_l = -w_l \otimes w_k, \quad k \neq l$$



Clifford module

The algebra homomorphism J :

$$J: \text{Cl}(W, \langle \cdot, \cdot \rangle_W) \rightarrow \text{End}(V)$$

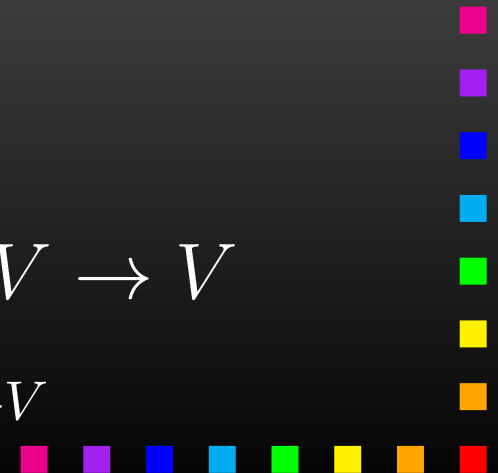
is called representation and (V, J) is

Clifford **module** for $\text{Cl}(W, \langle \cdot, \cdot \rangle_W)$

$$w \mapsto J(w, \cdot): V \rightarrow V$$

$$w \otimes w \mapsto J \circ J = J^2(w, \cdot): V \rightarrow V$$

$$-\langle w, w \rangle_W \mathbb{I} \mapsto -\langle w, w \rangle_W \text{Id}_V$$



Relation to Clifford module

$$(1) \quad \langle J(z, v), J(z, v') \rangle_V = \langle z, z \rangle_{\mathcal{Z}} \langle v, v' \rangle_V.$$

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

$$(3) \quad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V$$

Question: given (3) can we construct a general H -type algebra?

$$(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$$



Relation to Clifford module

$$(1) \quad \langle J(z, v), J(z, v') \rangle_V = \langle z, z \rangle_{\mathcal{Z}} \langle v, v' \rangle_V.$$

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

$$(3) \quad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V$$

Question: given (3) can we construct a general H -type algebra?

$$(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$$

Answer: **yes if we add (1) or (2)**



Relation to Clifford module

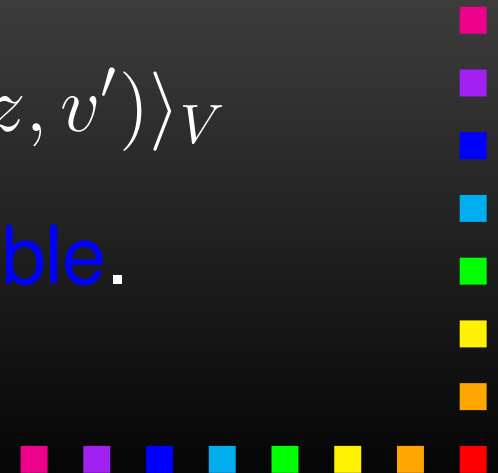
If for Clifford $Cl(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$ -module V

$$\left((3) \quad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V \right)$$

there is a scalar product $\langle \cdot, \cdot \rangle_V$ such that

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

hold, then $(V, \langle \cdot, \cdot \rangle_V)$ is called **admissible**.



Relation to Clifford module

If for Clifford $Cl(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$ -module V

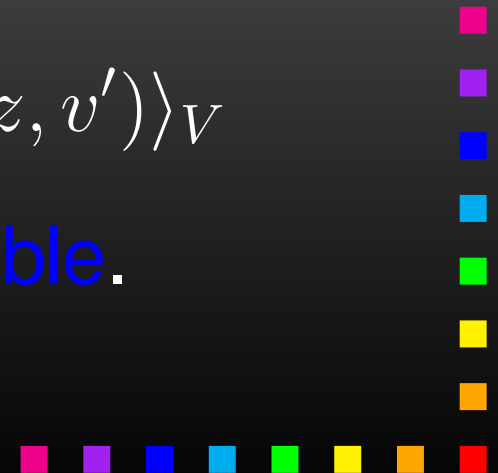
$$\left((3) \quad J^2(z, \cdot) = -\langle z, z \rangle_{\mathcal{Z}} \text{Id}_V \right)$$

there is a scalar product $\langle \cdot, \cdot \rangle_V$ such that

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V$$

hold, then $(V, \langle \cdot, \cdot \rangle_V)$ is called **admissible**.

Do admissible modules exist?



It is known

Given normalised composition

$$\langle \mu(u, v), \mu(u, v) \rangle_V = \langle u, u \rangle_U \langle v, v \rangle_V$$

the skew symmetric map $J(z, v) = \mu|_{\mathcal{Z} \times V}$ defines the representation of algebra $\text{Cl}(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$ on V .

Given an admissible $\text{Cl}(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$ – module V :

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V, \quad \text{for all } z \in \mathcal{Z}$$

there is $U = \mathcal{Z} \oplus U_0$ such that composition holds.



Existence of adm. module

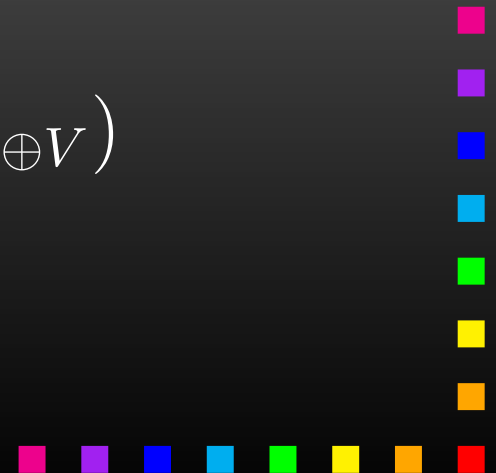
Given a $\text{Cl}(\mathcal{Z}, \langle \cdot, \cdot \rangle_{\mathcal{Z}})$ -module V , then V or $V \oplus V$ can be equipped with a scalar product satisfying

$$(2) \quad \langle J(z, v), v' \rangle_V = -\langle v, J(z, v') \rangle_V, \quad \text{for all } z \in \mathcal{Z}$$

P. Ciatti, 2000. Moreover

$$(V, \langle \cdot, \cdot \rangle_V) \quad \text{or} \quad (V \oplus V, \langle \cdot, \cdot \rangle_{V \oplus V})$$

is a neutral space



Classical H -type algebras

For $(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], (\cdot, \cdot))$ there is a basis $V = \text{span}\{v_{\alpha}\}$ and $\mathcal{Z} = \text{span}\{z_j\}$ such that

$$[v_{\alpha}, v_{\beta}] = \sum_j C_{\alpha\beta}^j z_j, \quad C_{\alpha\beta}^j \in \mathbb{Z}.$$

G. Crandall, J. Dodziuk, *Integral structures on H-type Lie algebras*, J. Lie Theory 12 (2002), no. 1, 69-79.

P. Eberlein, *Geometry of 2-step nilpotent Lie groups, Modern dynamical systems and applications*, Cambridge Univ. Press, Cambridge (2004), 67–101.

A. I. Mal'cev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation 39, 1951; Izv. Akad. Nauk USSR, Ser. Mat. 13 (1949), 9-32.



General H -type algebras

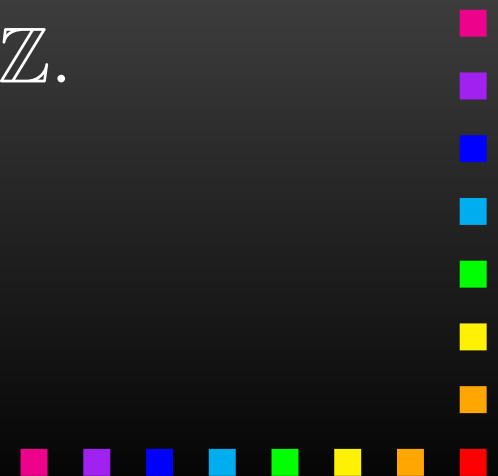
Do the general H -type algebras

$$(\mathcal{N} = V \oplus_{\perp} \mathcal{Z}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$$

admit integer constants?

$$[v_{\alpha}, v_{\beta}] = \sum_j C_{\alpha\beta}^j z_j, \quad C_{\alpha\beta}^j \in \mathbb{Z}.$$

Answer is **YES!**



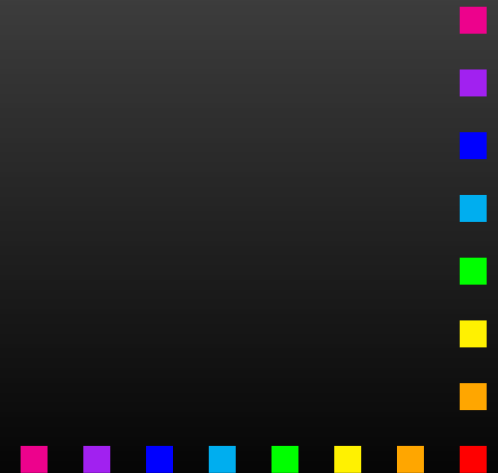
Isomorphisms preserving admissibility

$$\text{Cl}_{r+8,s} \sim \text{Cl}_{r,s} \otimes \text{Cl}_{8,0} \sim \text{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\text{Cl}_{r,s+8} \sim \text{Cl}_{r,s} \otimes \text{Cl}_{0,8} \sim \text{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\text{Cl}_{r+4,s+4} \sim \text{Cl}_{r,s} \otimes \text{Cl}_{4,4} \sim \text{Cl}_{r,s} \otimes \mathbb{R}(16)$$

$$\text{Cl}_{r,s+1} \sim \text{Cl}_{s,r+1}$$



Main idea of the proof

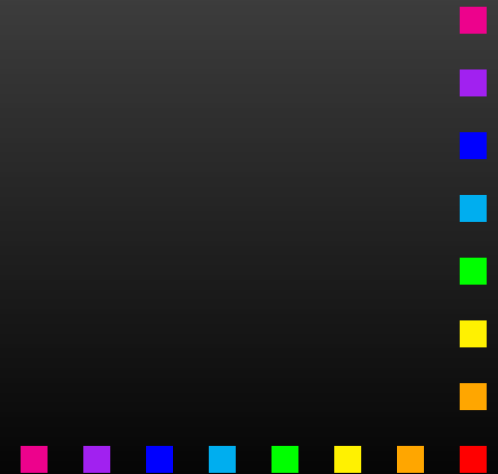
$$[v_\alpha, v_\beta] = \sum_j C_{\alpha\beta}^j z_j, \quad v_\alpha \in V, z \in \mathcal{Z}.$$

$$J: \mathcal{Z} \rightarrow \text{End}(V)$$

$$J_{z_j} v_\alpha = \sum_\beta B_{\alpha\beta}^j v_\beta.$$

$$\langle J_{z_j} v_\alpha, v_\beta \rangle_V = \langle [v_\alpha, v_\beta], z_j \rangle_{\mathcal{Z}}$$

$$C_{\alpha\beta}^j = B_{\alpha\beta}^j \nu_j^{\mathcal{Z}} \nu_\beta^V$$



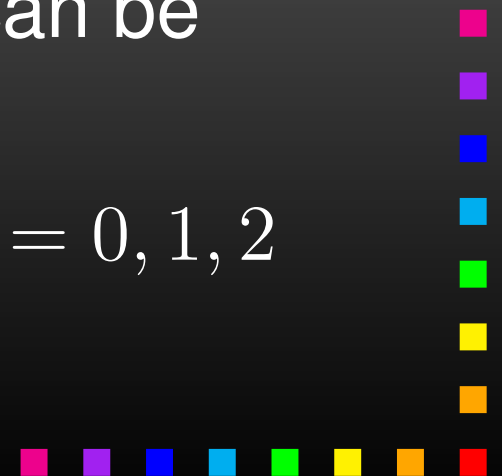
Main idea of the proof

$v_1 = w, v_2 = J_{z_1}w, \dots, v_{n+1} = J_{z_n}w, \quad n = \dim \mathcal{Z}$
is an orthogonal basis and since

$$\langle J_{z_j}v_\alpha, v_\beta \rangle_V = 0, \pm 1$$

we need to include $J_{z_i}J_{z_j}w$. But they can be non-orthogonal to v_1, \dots, v_{n+1} .

If $m = \dim V$, then $m = 2^{\mu+r}$, where $\mu = 0, 1, 2$
and $2r = n, n - 1, n - 2, n - 3, r \in \mathbb{N}$.



Main idea of the proof

Find operators $P_k: V \rightarrow V$ such that

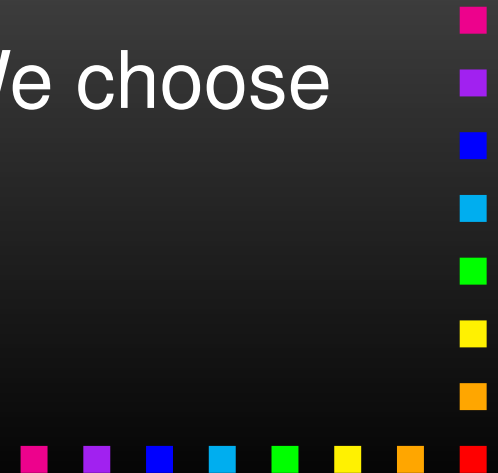
$$1. P_k^2 = \text{Id}_V \quad \longrightarrow \quad P_k w = w$$

$$2. \langle P_k v, P_k v \rangle_V = \langle v, v \rangle_V$$

$$3. P_k P_l = P_l P_k$$

Typical operator is $P = J_{z_i} J_{z_j} J_{z_l} J_{z_m}$. We choose $w \in V$

$$P_k w = w \quad \text{for all } k$$



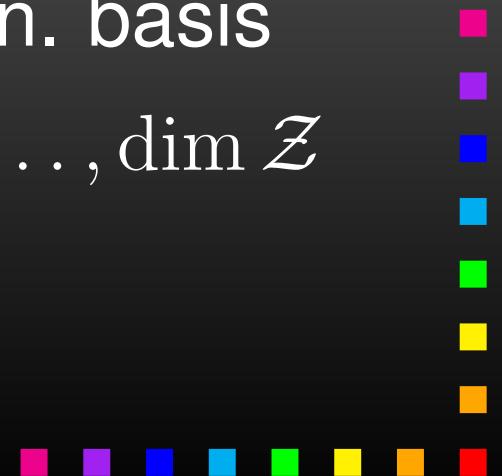
Main idea of the proof

We need such operators P_k to be sure that the following holds

a. $\langle w, w \rangle_V = 1$

b. $\{w, J_{z_i}w, J_{z_i}J_{z_j}w, J_{z_i}J_{z_j}J_{z_l}w, J_{z_i}J_{z_j}J_{z_l}J_{z_m}w\},$
 $1 \leq i < j < l < m \leq \dim V$ is an o.n. basis

c. J_{z_i} permute the basis for all $i = 1, \dots, \dim \mathcal{Z}$



Example $Cl_{0,6} \sim \mathbb{H}(4)$

Involution \ Generator	J_1	J_2	J_3	J_4	J_5	$J_6,$
$P_1 = J_1 J_2 J_3 J_4$	a	a	a	a	c	c
$P_2 = J_1 J_2 J_5 J_6$	a	a	c	c	a	a

Inv.	Eigenvalues			
P_1	+1		-1	
P_2	+1	-1	+1	-1
Eigenv.	$w, J_1 J_2 w$ $J_1 J_3 J_5 w, J_1 J_3 J_6 w$	$J_1 J_3 w, J_1 J_4 w$ $J_5 w, J_6 w$	$J_1 J_5 w, J_1 J_6 w$ $J_3 w, J_4 w$	$J_3 J_5 w, J_3 J_6 w$ $J_1 w, J_2 w$



Example $Cl_{0,6} \sim \mathbb{H}(4)$

Inv. \ Comp. op.	$J_1(+ \rightarrow -)$	$J_5(+ \rightarrow -)$	$J_2J_3J_5(+ \rightarrow -)$
P_1	a	c	c
P_2		a	c

Inv.	Eigenvalues			
P_1	+1		-1	
P_2	+1	-1	+1	-1
Eigenv.	w, J_1J_2w $J_1J_3J_5w, J_1J_3J_6w$	J_1J_3w, J_1J_4w J_5w, J_6w	J_1J_5w, J_1J_6w J_3w, J_4w	J_3J_5w, J_3J_6w J_1w, J_2w



Example $Cl_{0,6} \sim \mathbb{H}(4)$

$$J_1 w = J_2 J_3 J_4 w = J_2 J_5 J_6 w,$$

$$J_2 w = -J_1 J_3 J_4 w = -J_1 J_5 J_6 w,$$

$$J_3 w = J_1 J_2 J_4 w = -J_4 J_5 J_6 w,$$

$$J_4 w = -J_1 J_2 J_3 w = J_3 J_5 J_6 w,$$

$$J_5 w = J_1 J_2 J_6 w = -J_3 J_4 J_6 w,$$

$$J_6 w = -J_1 J_2 J_5 w = J_3 J_4 J_5 w,$$

$$J_1 J_3 J_5 w = -J_1 J_4 J_6 w = J_2 J_3 J_6 w = J_2 J_4 J_5 w,$$

$$J_1 J_3 J_6 w = J_1 J_4 J_5 w = -J_2 J_3 J_5 w = J_2 J_4 J_6 w.$$

$$J_1 J_2 w = -J_3 J_4 w = -J_5 J_6 w,$$

$$J_1 J_3 w = J_2 J_4 w,$$

$$J_1 J_4 w = -J_2 J_3 w,$$

$$J_1 J_5 w = J_2 J_6 w,$$

$$J_1 J_6 w = -J_2 J_5 w,$$

$$J_3 J_5 w = -J_4 J_6 w,$$

$$J_3 J_6 w = J_4 J_5 w.$$



The end

