Recovering an homogeneous polynomial from moments of its level set

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- Motivation
- An important property of Positively Homogeneous Functions (PHF)
- Some properties (convexity, polarity)
- Sub-level sets of minimum volume containing **K**
- Exact reconstruction from moments

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Exact reconstruction

Reconstruction of a shape $\mathbf{K} \subset \mathbb{R}^n$ (convex or not)

from knowledge of finitely many moments

$$y_\alpha = \int_{\mathbf{K}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, dx, \qquad \alpha \in \mathbb{N}_d^n,$$

for some integer d, is a difficult and challenging problem!

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from $y = (y_\alpha)$, $\alpha \in \mathbb{N}_d^n$, is even more difficult and challenging!

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Exact recovery (continued)

Examples of exact recovery:

- Quadrature (planar) Domains in ($\mathbb{R}^2$) (Gustafsson, He, Milanfar and Putinar (Inverse Problems, 2000))
  • via an exponential transform
- Convex Polytopes (in $\mathbb{R}^n$) (Gravin, Lasserre, Pasechnik and Robins (Discrete & Comput. Geometry (2012))
  • Use Brion-Barvinok-Khovanski-Lawrence-Pukhlikov moment formula for projections $\int_P \langle c, x\rangle^j \, dx$ combined with a Prony-type method to recover the vertices of P.
- and extension to Non convex poyhedra by Pasechnik et al.
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Approximate recovery can de done in multi-dimensions

(Cuyt, Golub, Milanfar and Verdonk, 2005) via :

- (multi-dimensional versions of) homogeneous Padé approximants applied to the Stieltjes transform.
- cubature formula at each point of grid
- solving a linear system of equations to retrieve the indicator function of $\mathbf{K}$

This talk

- Exact recovery.
- $\mathbf{K} = \{ x \in \mathbb{R}^n : g(\mathbf{x}) \leq 1 \}$ compact.
- g is a nonnegative homogeneous polynomial
- Data are finitely many moments:

$$y_\alpha = \int_{\mathbf{K}} \mathbf{x}^\alpha \, d\mathbf{x}, \quad \alpha \in \mathbb{N}^n_d.$$

• Also works for Quasi-homogeneous polynomials, i.e., when

$$g(\lambda^{u_1} x_1, \ldots, \lambda^{u_n} x_n) = \lambda \, g(x), \qquad x \in \mathbb{R}^n,$$

for some vector $u \in \mathbb{Q}^n$.

(d-Homogeneous =u-quasi homogeneous with $u_i = \frac{1}{d}$ for all i).

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A little detour

Positively Homogeneous functions (PHF) form a wide class of functions encountered in many applications. As a consequence of homogeneity, they enjoy very particular properties, and among them the celebrated and very useful Euler's identity which allows to deduce additional properties of PHFs in various contexts.

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So we are now concerned with PHFs, their sublevel sets and in particular, the integral

$$y \mapsto I_{g,h}(y) := \int_{\{x \,:\, g(\mathbf{x}) \leq y\}} h(\mathbf{x})\, d\mathbf{x},$$

as a function $I_{g,h} : \mathbb{R}_+ \to \mathbb{R}$ when g, h are PHFs.

With y fixed, we are also interested in

$$g \mapsto I_{g,h}(y),$$

now as a function of g, especially when g is a nonnegative homogeneous polynomial.

Nonnegative homogeneous polynomials are particularly interesting as they can be used to approximate norms; see e.g. Barvinok

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Some motivation

Interestingly, the latter integral is related in a simple and remarkable manner to the non-Gaussian integral

$$\int_{\mathbb{R}^n} h \exp(-g) dx.$$

Functional integrals appear frequently in quantum Physics

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exact formulas for $\int \exp(-g)\, dx$, the most well-known being when $\deg g = 2$,

$$d = 2 \Rightarrow \int \exp(-g)\, dx \;=\; \frac{\mathrm{Cte}}{\sqrt{\det(g)}}.$$

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Observe that $\det(g)$ is an algebraic invariant of g, The key tools are discriminants and $SL(n)$-invariants.

An integral

$$J(g) := \int \exp(-g)\,dx$$

is called a discriminant integral.

Integral discriminants satisfy WARD Identities

$$\left(\frac{\partial}{\partial g_{a_1\cdots a_n}}\frac{\partial}{\partial g_{b_1\cdots b_n}} - \frac{\partial}{\partial g_{c_1\cdots c_n}}\frac{\partial}{\partial g_{d_1\cdots d_n}}\right)\cdot J(g) = 0,$$

where $a_i + b_i = c_i + d_i$ for all i.

which permits to obtain exact formulas in low-dimensional cases in terms of algebraic invariants of g. See e.g. Morosov and Shakirov[1]

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In particular, as a by-product in the important particular case when $h = 1$, they have proved that for all *forms* g of degree d,

$$\begin{aligned} \mathrm{Vol}\left(\{x \,:\, g(x) \leq 1\}\right) &= \int_{\{x\,:\,g(x)\leq 1\}} dx \\ &= \mathrm{cte}(d) \cdot \int_{\mathbb{R}^n} \exp(-g) d\mathbf{x}, \end{aligned}$$

where the constant depends only on d and n.

In fact, a formula of exactly the same flavor was already known for convex sets, and was the initial motivation of our work. Namely, if $C \subset \mathbb{R}^n$ is convex, its support function

$$x \mapsto \sigma_C(x) := \sup \{x^T y \,:\, y \in C\},$$

is a PHF of degree 1, and the polar $C^\circ \subset \mathbb{R}^n$ of C is the convex set $\{x \,:\, \sigma_C(x) \leq 1\}$.

Then . . .

$$\mathrm{vol}\,(C^\circ) = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-\sigma_C(x))\, dx, \qquad \forall C.$$

I. An important property of PHF's

Theorem

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable mapping, and let $g \geq 0$ and h be PHFs of respective degree $0 \neq d, p \in \mathbb{Z}$ and such that $\int |h| \exp(-g) dx$ is finite,

$$\int_{\mathbb{R}^n} \phi(g(x))\, h(x)\, dx \;=\; C(\phi, d, p) \cdot \int_{\mathbb{R}^n} h \exp(-g)\, dx,$$

where the constant $C(\phi, d, p)$ depends only on ϕ, d, p.
In particular, if the sublevel set $\{x \,:\, g(x) \leq 1\}$ is bounded, then

$$\int_{\{x : g(x) \leq y\}} h\, dx \;=\; \frac{y^{(n+p)/d}}{\Gamma(1 + (n+p)/d)} \int_{\mathbb{R}^n} h \exp(-g)\, dx,$$

with Γ being the standard Gamma function

Proof for nonnegative h

With $z = (z_1, \ldots, z_{n-1})$, do the change of variable $x_1 = t$, $x_2 = t\,z_1, \ldots, x_n = t\,z_{n-1}$) so that one may decompose $\int_{\mathbb{R}^n} \phi(g(x))\, h(x) dx$ into the sum

$$
\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(1,z))\, h(1,z)\, dt\, dz \\
+ & \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(-1,-z))\, h(-1,-z)\, dt\, dz, \\
= & \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty t^{n+p-1} \phi(t^d g(1,z))\, dt \right) h(1,z)\, dz \\
+ & \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty t^{n+p-1} \phi(t^d g(-1,-z))\, dt \right) h(-1,-z)\, dz,
\end{aligned}
$$

where the last two integrals are obtained from the sum of the previous two by using Tonelli's Theorem.

Proof (continued)

Next, with the change of variable $u = t\, g(1, z)^{1/d}$ and $u = t\, g(-1, -z)^{1/d}$

$$\int_{\mathbb{R}^n} \phi(g(x))\, h(x)\, dx \;=\; \underbrace{\left(\int_{\mathbb{R}_+} u^{n+p-1}\, \phi(u^d)\, du\right)}_{\mathrm{Cte}(\phi,p,d)} \cdot A(g,h),$$

with

$$A(g,h) = \int_{\mathbb{R}^{n-1}} \left(\frac{h(1,z)}{g(1,z)^{(n+p)/d}} + \frac{h(-1,-z)}{g(-1,-z)^{(n+p)/d}} \right) dz.$$

□

Choosing $\phi(t) = \exp(-t)$ on $[0, +\infty)$ yields:

$$\int_{\mathbb{R}^n} \exp(-g(x))\, h(x)\, dx \;=\; \frac{\Gamma(1+(n+p)/d)}{n+p} \cdot A(g,h),$$

whereas, choosing $\phi(t) = \mathrm{I}_{[0,1]}(t)$ on $[0, +\infty)$ yields:

$$\int_{\{x\,:\,g(x)\leq 1\}} h(x)\, dx \;=\; \frac{1}{n+p} \cdot A(g,h),$$

And so in particular, whenever g is nonnegative and $\{x \,:\, g(x) \leq 1\}$ is bounded:

Theorem

If g, h are PHFs of degree $0 < d$ and p respectively, then:

$$\int_{\{x\,:\,g(x)\leq y\}} h\,dx \quad = \quad \frac{y^{(n+p)/d}}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} \exp(-g)\,h\,dx$$

$$\mathrm{vol}\,(\{x \,:\, g(x) \leq y\}) \quad = \quad \frac{y^{n/d}}{\Gamma(1+n/d)} \int_{\mathbb{R}^n} \exp(-g)\,dx$$

An alternative proof

Let g, h be nonnegative so that $I_{g,h}(y)$ vanishes on $(-\infty, 0]$. Its Laplace transform $\lambda \mapsto \mathcal{L}_{I_{g,h}}(\lambda) = \int_0^\infty \exp(-\lambda y) I_{g,h}(y)\, dy$

$$
\begin{aligned}
\mathcal{L}_{I_{g,h}}(\lambda) &= \int_0^\infty \exp(-\lambda y) \left(\int_{\{x: g(x) \leq y\}} h dx \right) dy \\
&= \int_{\mathbb{R}^n} h(x) \left(\int_{g(x)}^\infty \exp(-\lambda y) dy \right) dx \quad \text{[by Fubini]} \\
&= \frac{1}{\lambda} \int_{\mathbb{R}^n} h(x) \exp(-\lambda g(x))\, dx \\
&= \frac{1}{\lambda^{1+(n+p)/d}} \int_{\mathbb{R}^n} h(z) \exp(-g(z))\, dz \quad \text{[by homog]} \\
&= \frac{\int_{\mathbb{R}^n} h(z) \exp(-g(z))\, dz}{\Gamma(1 + (n+p)/d)}\, \mathcal{L}_{y^{(n+p)/d}}(\lambda).
\end{aligned}
$$

And so, by uniqueness of the Laplace transform,

$$I_{g,h}(y) = \frac{y^{(n+p)/d}}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} h(x) \exp(-g(x))\, dx,$$

II. Approximating a non gaussian integral

Hence computing the non Gaussian integral $\int \exp(-g)\,dx$

reduces to computing the volume of the level set
$G := \{x \,:\, g(x) \,\leq\, 1\}$,

... which is the same as solving the optimization problem:

$$\begin{array}{rll} \max_{\mu} & \mu(G) & \\ \text{s.t.} & \mu + \nu & = \lambda \\ & \mu(\mathbf{B} \setminus G) & = 0 \end{array}$$

where :

- $\mathbf{B}$ is a box $[-a, a]^n$ containing G and
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. . . and we know how to approximate $\mu(G)$ as closely as desired

Let $G \subseteq \mathbf{B} := [-1,1]^n$ (possibly after scaling), and let $z = (z_\alpha)$, $\alpha \in \mathbb{N}^n_{2k}$, be the moments of the Lebesgue measure λ on B.

Solve the hierarchy of semidefinite programs:

$$\begin{array}{rl} \rho_k = \max & y_0 \\ \text{s.t.} & \mathbf{M}_k(y), \mathbf{M}_k(v) \succeq 0, \\ & \mathbf{M}_{k-\lceil(d)/2\rceil}(g\,y) \succeq 0 \\ & \mathbf{M}_{k-1}((1-x_i^2)\,v) \succeq 0, \quad i = 1,\ldots,n \\ & y_\alpha + v_\alpha = z_\alpha, \quad \alpha \in \mathbb{N}^n_{2k} \end{array}$$

for some moment and localizing matrices $\mathbf{M}_k(y)$ and $\mathbf{M}_k(g,y)$.

- The linear constraints $y_\alpha + v_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}^n_{2k}$ "ensure" $\mu + \nu = \lambda$, while the "$\succeq 0$" constraints "ensure" $\text{supp}\,\mu = G$ and $\text{supp}\,\nu = \mathbf{B}$.

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Lemma

$\rho_k \to \lambda(G)$ *as* $k \to \infty$.
(see: Approximate volume and integration for basic semi algebraic sets*, Henrion, Lasserre and Savorgnan, SIAM Review 51, 2009.)*

However . . .

the resulting SDPs are numerically difficult to solve.

Solving the dual reduces to approximating the indicator function $I(G)$ by polynomials of increasing degrees $\to$ Gibbs effect, etc.

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Another identity

Corollary

If g has degree d and G is bounded then

$$\frac{\displaystyle\int_{\{x\,:\,g(x)\leq y\}} \exp(-g)\,dx}{\displaystyle\int_{\mathbb{R}^n} \exp(-g)\,dx} = \frac{\displaystyle\int_0^y t^{n/d-1}\,\exp(-t)dt}{\displaystyle\int_0^\infty t^{n/d-1}\,\exp(-t)dt} = \frac{\displaystyle\int_0^y t^{n/d-1}\,\exp(-t)dt}{\Gamma(n/d)}$$

expresses how fast $\mu(\{x\,:\,g(x)\leq y\})$ goes to $\mu(\mathbb{R}^n)$ as $y\to\infty$, for the Borel measure $d\mu=\exp(-g)\,dx$.

It is like for the Gamma function $\Gamma(n/d)$ when approximated by $\int_0^y t^{n/d-1}\exp(-t)dt$.

An interesting issue is to analyze how the Lebesgue volume $\mathrm{vol}\,\{x \in \mathbb{R}^n \,:\, g(x) \,\leq\, 1\}$, (i.e. $\mathrm{vol}\,(G)$) changes with g.
Recall that $G = \{x : g(x) \leq 1\}$.

Corollary

Let h be a PHF of degree p and let $C_d \subset \mathbb{R}[x]_d$ be the convex cone of homogeneous polynomials of degree at most d such that G is bounded. Then the function $f_h : C_d \to \mathbb{R}$,

$$g \mapsto f_h(g) \,:=\, \int_G h\,dx, \qquad g \in C_d,$$

- *is a PHF of degree $-(n+p)/d$,*
- *convex whenever h is nonnegative and strictly convex if $h > 0$ on $\mathbb{R}^n \setminus \{0\}$*

Convexity (continued)

Corollary (continued)

Moreover, if h is continuous and $\int |h| \exp(-g)\,dx < \infty$ then:

$$\frac{\partial f_h(g)}{\partial g_\alpha} = \frac{-1}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} x^\alpha\, h \exp(-g)\,dx$$

$$= \frac{-\Gamma(2+(n+p)/d)}{\Gamma(1+(n+p)/d)} \int_G x^\alpha\, h\,dx$$

$$\frac{\partial^2 f_h(g)}{\partial g_\alpha \partial g_\beta} = \frac{-1}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} x^{\alpha+\beta}\, h \exp(-g)\,dx$$

PROOF: Just use

$$\int_{\{x\,:\,g(x)\leq 1\}} h\,dx \;=\; \frac{1}{\Gamma(1+(n+p)/d)}\int_{\mathbb{R}^n} h\,\exp(-g)\,dx$$

Notice that proving convexity directly would be non trivial but becomes easy when using the previous lemma!

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III. Polarity

For a set $C \subset \mathbb{R}^n$, recall:

- The support function $x \mapsto \sigma_C(x) := \sup_y \{x^T y \,:\, y \in C\}$
- The POLAR $C^\circ := \{x \in \mathbb{R}^n \,:\, \sigma_C(x) \leq 1\}$
- and for a PHF g of degree d, its Legendre-Fenchel conjugate $g^*(x) = \sup_y \{x^T y - g(y)\}$ is a PHF of degree q with $\frac{1}{d} + \frac{1}{q} = 1$.

Polarity (continued)

Lemma

Let g be a closed proper convex PHF of degree $1 < d$ and let $G = \{x : g(x) \leq 1/d\}$. Then:

$$\begin{aligned} G^{\circ} &= \{x \in \mathbb{R}^n \,:\, g^*(x) \,\leq\, 1/q\} \\ \mathrm{vol}\,(G) &= \frac{p^{-n/p}}{\Gamma(1+n/p)} \int \exp(-g)\,dx \\ \mathrm{vol}\,(G^{\circ}) &= \frac{q^{-n/q}}{\Gamma(1+n/q)} \int \exp(-g^*)\,dx \end{aligned}$$

$\rightarrow$ yields completly symmetric formulas for g and its conjugate g^*.

Examples

- $g(x) = |x|^3$ so that $g^*(x) = \frac{2}{3\sqrt{3}}|x|^{3/2}$. And so

$$G = [-3^{-1/3}, 3^{-1/3}]; \quad G^\circ = [-3^{1/3}, 3^{1/3}].$$

- TV screen: $g(x) = x_1^4 + x_2^4$ so that $g^*(x) = 4^{-4/3}3(x_1^{4/3} + x_2^{4/3})$. And,

$$G = \{x : x_1^2 + x_2^4 \leq \frac{1}{4}\}; \quad G^\circ = \{x : x_1^{4/3} + x_2^{4/3} \leq 4^{1/3}\}.$$

- $g(x) = |x|$ so that $d \not> 1$, and $g^*(x) = 0$ if $x \in [-1, 1]$, and $+\infty$ otherwise. Hence $G = \{x : |x| \leq 1\} = [-1, 1]$ and with $q = +\infty$,

$$G^\circ = [-1, 1] = \{x : g^*(x) \leq \frac{1}{q} = 0\}.$$

IV. A variational property of homogeneous polynomials

Let $\mathbf{v}_d(x)$ be the vector of monomials (x^α) of degree d, i.e., such that $\alpha_1 + \cdots + \alpha_n = d$.
If $g \in \mathbb{R}[x]_{2d}$ is homogeneous and SOS then

$$g(x) = \frac{1}{2}\mathbf{v}_d(x)^T \Sigma \mathbf{v}_d(x),$$

for some real symmetric positive semidefinite matrix $\Sigma \succeq 0$.

And if $d = 1$ one has the Gaussian property

$$\int_{\mathbb{R}^n} \exp(-g)\, dx = \frac{(2\pi)^{n/2}}{\sqrt{\det \Sigma}},$$

$$\frac{\displaystyle\int_{\mathbb{R}^n} \mathbf{v}_d(x)\, \mathbf{v}_d(x)^T \exp(-g)\, dx}{\displaystyle\int_{\mathbb{R}^n} \exp(-g)\, dx} = \Sigma^{-1}.$$

In other words, if μ is the Gaussian measure

$$\mu(B) := \frac{\displaystyle\int_B \exp\left(-\frac{1}{2}x^T\Sigma\,x\right)\,dx}{\displaystyle\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^T\Sigma\,x\right)\,dx}, \quad \forall B,$$

then its (covariance) matrix of moments of order 2 satisfies:

$$\mathbf{M}_1(\Sigma) := \int_{\mathbb{R}^n} x\,x^T\,d\mu(x) = \Sigma^{-1}.$$

... not true anymore for $d > 1$!

However, let $\ell(d) = \binom{n+d}{n-1}$, and $\mathcal{S}_{++}^{\ell(d)}$ be the cone of real positive definite $\ell(d) \times \ell(d)$ matrices. Let $k := n/(2d\ell(d))$.

With $\Sigma \in \mathcal{S}_{++}^{\ell(d)}$, define the probability measure μ

$$\mu(B) := \frac{\displaystyle\int_B \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx}{\displaystyle\int_{\mathbb{R}^n} \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx}, \quad \forall B,$$

with matrix of moments of order $2d$ given by:

$$\mathbf{M}_d(\Sigma) := \int_{\mathbb{R}^n} \mathbf{v}_d(x)\, \mathbf{v}_d(x)^T\, d\mu(x).$$

Define $\theta_d : \mathcal{S}_{++}^{\ell(d)} \to \mathbb{R}$ to be the function

$$\Sigma \mapsto \theta_d(\Sigma) := (\det \Sigma)^k \int_{\mathbb{R}^n} \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx.$$

Theorem

$$\mathbf{M}_d(\Sigma) = \Sigma^{-1} \iff \nabla\theta_d(\Sigma) = 0$$

Hence critical points Σ^* *of* θ_d *have the Gaussian property*

$$\frac{\int \mathbf{v}_d(x)\mathbf{v}_d(x)^T \exp\left(-k\mathbf{v}_d(x)^T \Sigma^*\, \mathbf{v}_d(x)\right)\, dx}{\int \exp\left(-k\mathbf{v}_d(x)^T \Sigma^*\, \mathbf{v}_d(x)\right)\, dx} = (\Sigma^*)^{-1}$$

$\star$ If $d = 1$ then $\theta_d(\cdot)$ is constant and so $\nabla\theta_d(\cdot) = 0$.
$\star$ If $d > 1$ then $\theta_d(\cdot)$ is constant in each ray $\lambda\Sigma$, $\lambda > 0$.

Proof

$$
\begin{aligned}
\nabla\theta_d(\Sigma) &= k\frac{\Sigma^{\mathbb{A}}}{\det\Sigma}\,\theta_d(\Sigma) \\
&\quad -k(\det\Sigma)^k\int_{\mathbb{R}^n}\mathbf{v}_d(x)\mathbf{v}_d(x)^T\exp\left(-k\mathbf{v}_d(x)^T\Sigma\,\mathbf{v}_d(x)\right)\,dx \\
&= k\theta_d(\Sigma)\left[\Sigma^{-1}-\mathbf{M}_d(\Sigma)\right]
\end{aligned}
$$

and so

$$
\mathbf{M}_d(\Sigma) = \Sigma^{-1} \quad \Rightarrow \quad \nabla\theta_d(\Sigma) = 0.
$$

V. Sublevel sets G of minimum volume

If $\mathbf{K} \subset \mathbb{R}^n$ is compact then computing the ellipsoid ξ of minimum volume containing $\mathbf{K}$ is a classical problem whose optimal solution is called the Löwner-John ellipsoid.
So consider the following problem:

Find an homogeneous polynomial $g \in \mathbb{R}[x]_{2d}$ such that its sub level set $G := \{x \,:\, g(x) \leq 1\}$ contains $\mathbf{K}$ and has minimum volume among all such levels sets with this inclusion property.

Let $\mathbf{P}[x]_{2d}$ be the convex cone of homogeneous polynomials of degree $2d$ with compact sub-level set $\{x : g(x) \leq 1\}$, and with $\mathbf{K} \subset \mathbb{R}^n$, let $C_{2d}(\mathbf{K})$ be the convex cone of polynomials nonnegative on $\mathbf{K}$.

Lemma

Let $\mathbf{K} \subset \mathbb{R}^n$ be compact. The minimum volume of a sublevel set $\{\mathbf{x} : g(\mathbf{x}) \leq 1\}$, $g \in \mathbf{P}[x]_{2d}$, that contains $\mathbf{K} \subset \mathbb{R}^n$ is $\rho/\Gamma(1 + n/2d)$ where:

$$\mathcal{P} : \qquad \rho = \inf_{g \in \mathbf{P}[x]_{2d}} \left\{ \int_{\mathbb{R}^n} \exp(-g)\, dx \;:\; 1 - g \in C_{2d}(\mathbf{K}) \right\}.$$

a finite-dimensional convex optimization problem!

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Proof

• We have seen that:

$$\mathrm{vol}\,(\{x\,:\,g(x)\leq 1\})\,=\,\frac{1}{\Gamma(1+n/2d)}\int_{\mathbb{R}^n}\exp(-g)\,dx.$$

Moreover, the sub-level set $\{x\,:\,g(x)\leq 1\}$ contains **K** if and only if $1-g\in C_{2d}(\mathbf{K})$, and so $\rho/\Gamma(1+n/2d)$ is the minimum value of all volumes of sub-levels sets $\{x\,:\,g(x)\leq 1\}$, $g\in\mathbf{P}[\mathbf{x}]_{2d}$, that contain **K**.

• Now since $g\mapsto\int_{\mathbb{R}^n}\exp(-g)dx$ is strictly convex and $C_{2d}(\mathbf{K})$ is a convex cone, problem $\mathcal{P}$ is a finite-dimensional convex optimization problem. $\square$

V (continued). Characterizing an optimal solution

Theorem

(a) $\mathcal{P}$ *has a unique optimal solution* $g^* \in \mathbf{P}[x]_{2d}$ *and there exists a Borel measure* μ^* *supported on* $\mathbf{K}$ *such that:*

$$(*) : \quad \left\{ \begin{array}{l} \displaystyle\int_{\mathbb{R}^n} x^\alpha \exp(-g^*)dx \; = \; \int_{\mathbf{K}} x^\alpha \, d\mu^*, \quad \forall |\alpha| = 2d \\ \displaystyle\int_{\mathbf{K}} (1 - g^*) \, d\mu^* = 0 \end{array} \right.$$

In particular, μ^* *is supported on the real variety* $V := \{x \in \mathbf{K} : g^*(\mathbf{x}) = 1\}$ *and in fact,* μ^* *can be substituted with another measure* ν^* *supported on at most* $\binom{n+2d-1}{2d} + 1$ *points of* V.

(b) *Conversely, if* $g^* \in \mathbf{P}[x]_{2d}$ *and* μ^* *satisfy (*) then* g^* *is an optimal solution of* $\mathcal{P}$.

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VI. Recovering g from moments of G

Write $g(x) = \sum_\beta g_\beta \, x^\beta$.

Lemma

If g is nonnegative and d-homogeneous with G compact then:

$$\underbrace{\int_G x^\alpha \, g(x) \, dx}_{\sum_\beta g_\beta \, y_{\alpha+\beta}}, = \frac{n + |\alpha|}{n + d + |\alpha|} \underbrace{\int_G \mathbf{x}^\alpha \, dx}_{y_\alpha}, \qquad \alpha \in \mathbb{N}^n.$$

and so we see that the moments (y_α) satisfy linear relationships explicit in terms of the coefficients of the polynomial g that describes the boundary of G.

So let us write $\mathbf{g} \in \mathbb{R}^{s(d)}$ the unknown vector of coefficients of the unknown polynomial g.
Let $\mathbf{M}_d(y)$ be the moment matrix of order d whose rows and columns are indexed in the canonical basis of monomials (x^α), $\alpha \in \mathbb{N}^n_d$, and with entries

$$\mathbf{M}_d(y)(\alpha, \beta) = y_{\alpha+\beta}, \qquad \alpha, \beta \in \mathbb{N}^n_d.$$

and let $\mathbf{y}^d$ be the vector (y_α), $\alpha \in \mathbb{N}^n_d$.

Previous Lemma states that

$$\mathbf{M}_d(y)\,\mathbf{g} = \mathbf{y}^d,$$

or, equivalently,

$$\mathbf{g} = \mathbf{M}_d(y)^{-1}\,\mathbf{y}^d,$$

because the moment matrix $\mathbf{M}_d(y)$ is nonsingular whenever G has nonempty interior.

In other words ...

one may recover g EXACTLY from knowledge of moments (y_α) of order d and $2d$!

Conclusion

• Compact sub-level sets $G := \{x : g(x) \leq y\}$ of homogeneous polynomials exhibit surprising properties. E.g.:

- convexity of $\mathrm{volume}(G)$ with respect to the coefficients of g
- Integrating a PHF h on G reduce to evaluating the non Gaussian integral $\int h \exp(-g)dx$
- A variational property yields a Gaussian-like property
- exact recovery of G from finitely moments.

Also works for quasi-homogeneous polynomials with bounded sublevel sets!

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Practical and important issues

- COMPUTATION!: Efficient evaluation of $\int_{\mathbb{R}^n} \exp(-g)\,dx$, or equivalently, evaluation of $\mathrm{vol}\,(\{x : g(x) \leq 1\})$!

  • The property

$$\int_G \mathbf{x}^\alpha g(x)\,dx \;=\; \frac{n+|\alpha|}{n+d+|\alpha|} \int_G x^\alpha\,dx, \qquad \forall \alpha,$$

  helps a lot to improve efficiency of the method in Henrion, Lasserre and Savorgnan (SIAM Review)

- What about non-homogeneous polynomials?

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THANK YOU!