Affine Semigroups and Parametric Polyhedra with Prescribed Number of Lattice Points

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What is the problem?

Why do you care?

Coin exchange Problems

We wish to know, using USA coins (pennies, nickels, dimes and quarters)

1. **How many ways** are there to give change for b cents?
2. What is **the smallest number of coins necessary** to do so?
3. What is **largest quantity** b which cannot be expressed using these coins?

Holes Gaps and the Frobenius Problem

- Let $a = (a_1, \ldots, a_n)^T \in \mathbb{Z}^n_{>0}$ with $\gcd(a_i) = 1$. The values of coins!
- We study $\mathrm{sg}_a := \{b : b = a_1x_1 + a_2x_2 + \ldots a_nx_n, x_i \in \mathbb{Z}_+\}$
- Deciding whether $b \in \mathrm{sg}_a$ is an NP-complete problem. Counting solutions is #P-complete.
- We say b is a **gap** or a **hole** cannot be represented as a non-negative integral combination of the a_i's.
- *Classical Frobenius problem*: Find the largest number b which is a hole.
- When n is not fixed this is an NP-hard problem Ramirez Alfonsin (1996).
- For fixed n the Frobenius number can be computed in polynomial time Kannan (1992) Barvinok and Woods (2003).

Generalization!!

- **M. Beck and S. Robins (2004):** introduced the $< k$-**Frobenius number**: The largest right-hand number b representable in *no more than* $k - 1$ ways as a non-linear combination of the entries $a_1, a_2, \ldots, a_n$.
- They gave formulas for $n = 2$ of the $< k$-Frobenius number, but for general n and k only bounds on the $< k$-Frobenius number are available (**work by Aliev, Henk, Fushansky, etc**).
- For $\text{sg}_a := \{b : b = a_1x_1 + a_2x_2 + \ldots a_nx_n, x_i \in \mathbb{Z}_+\}$ we can ask
- For which b is there a **unique** way to give change?
- For which b are there **at most k ways** to give change?
- For which b are there **at least k ways** to give change?
- **Observation:** If one knows the solution of the $\geq k$ problem one can also solve the $< k$ problem and vice versa!!

The Question for General Semigroups

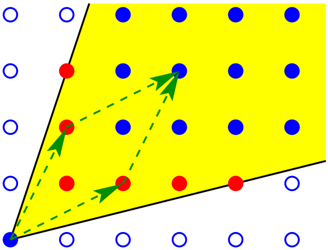
- Let $A \in \mathbb{Z}^{d \times n}$ and $b \in \mathbb{Z}^d$. Tthink of A as fixed and b is a parameter.
- We study **parametric family** of linear Diophantine problems $Ax = b$, $x \geq 0$, $x \in \mathbb{Z}^n$ (*).
- Let $P_A(b) = \{x : x \in \mathbb{R},\ Ax = b,\ x \geq 0\}$ be the convex polyhedron of *real solutions* of Problem (*)
- Let $IP_A(b) = P_A(b) \cap \mathbb{Z}^n$.
- Let $\mathrm{sg}(A)$ be the finitely generated semigroup all non-negative integer combinations of the columns of A,

$$\mathrm{sg}(A) = \{b : Ax = b, \text{for some } x \in \mathbb{Z}^n, x_i \geq 0\}.$$

- Let $\mathrm{cone}(A)$ the *cone generated by* A, i.e., the set of all non-negative real combinations of columns of A.

Finite generation of lattice points in Cones

(Gordan's lemma) Given a matrix A, let cone(A) and sg(A) be the cone and affine semigroup generated by A.
Moreover cone(A) $\cap \mathbb{Z}^d$ is finitely generated in terms of sg(A) in the sense that there exist finitely many $z_1, \ldots, z_u \in \Pi_A \cap \mathbb{Z}^d_+$ such that cone(A) $= \bigcup_{i=1}^{u} z_i + \text{sg}(A)$;
Those elements $z_1, \ldots, z_u$ are the famous *Hilbert bases*



- **Fact** $\mathrm{sg}(A)$ is not always equal to $\mathrm{cone}(A) \cap \mathbb{Z}^d$, but it is always contained in it.
- A **hole** a lattice point that is in $\mathrm{cone}(A)$ but not in $\mathrm{sg}(A)$!
- Surprisingly, the set of holes may be finite or infinite.
- There is a finite description of the holes in terms of finitely many generators.

Theorem (Hemmecke-Takemura-Yoshida)

There exists an algorithm that computes for an integral matrix A a finite explicit representation for the set H of holes of the semigroup Q generated by the columns of A, that is, the algorithm computes (finitely many) vectors $h_i \in \mathbb{Z}^n$ and monoids M_i, each given by a finite set of generators in $\mathbb{Z}^n$, $i \in I$, such that

$$H = \bigcup_{i \in I} \left(\{h_i\} + M_i \right).$$

Example of holes

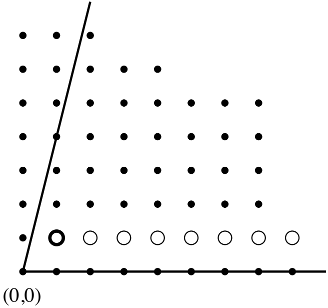
Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}$$

The cone has infinitely many holes H, but it is a finitely generated set!!

$$H = \{(1,1)^\intercal + \alpha \cdot (1,0)^\intercal : \alpha \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+$ denote the set of nonnegative integers.



QUESTIONS!!! *the fundamental problems of k-feasibility*

Let $IP_A(b) = \{x : Ax = b,\ x \geq 0,\ x \in \mathbb{Z}^n\}$ and $k \geq 1$ an integer.

- Are there *at least* k distinct solutions for $IP_A(b)$? If yes, we say that the problem is $\geq$ *k-feasible*.
- Are there *exactly* k distinct solutions for $IP_A(b)$? If yes, we say that the problem is $=$ *k-feasible*.
- Are there *less than* k distinct solutions for $IP_A(b)$? If yes, we say that the problem is $<$ *k-feasible*.
- Let $\text{sg}_{\geq k}(A)$ (respectively $\text{sg}_{=k}(A)$ and $\text{sg}_{<k}(A)$) be the set of right-hand side vectors $b \in \text{cone}(A) \cap \mathbb{Z}^d$ that make $IP_A(b)$ $\geq$ k-feasible (respectively $=$ k-feasible, $<$ k-feasible).
- **Note:** $\text{sg}(A) = \text{sg}_{\geq 1}(A)$, the holes of cone(A) are $\text{sg}_{<1}(A)$.

RESULTS

Structure of $\mathrm{sg}_{\geq k}(A)$ and $\mathrm{sg}_{<k}(A)$: Finite Generation

Theorem

(i) *There exists a monomial ideal* $I(A) \subset \mathbb{Q}[x_1, \ldots, x_n]$ *such that*

$$\mathrm{sg}_{\geq k}(A) = \{A\lambda : \lambda \in E(A)\}, \tag{1}$$

where $E(A)$ *is the set of exponents of monomials of* $I(A)$.

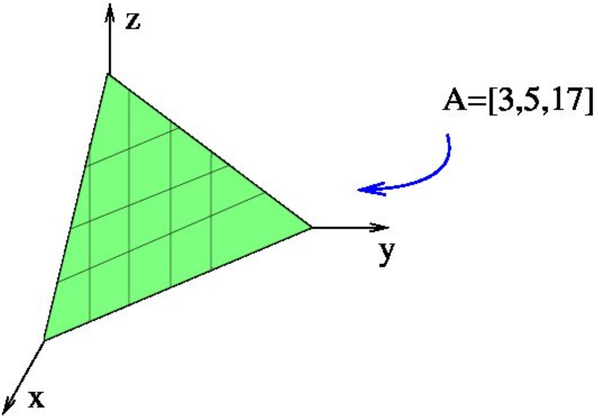
(ii) *We can compute (finitely many) vectors* $h_i \in \mathbb{Z}^n$ *and monoids* M_i*, each given by a finite set of generators in* $\mathbb{Z}^n$*,* $i \in I$*, such that*

$$\mathrm{sg}_{\geq k}(A) = \bigcup_{i \in I} (\{h_i\} + M_i).$$

(ii) *The set* $\mathrm{sg}_{<k}(A)$ *can be written as a finite union of translates of the sets* $\{A\lambda : \lambda \in S\}$*, where* S *is a coordinate subspace of* $\mathbb{Z}^n_{\geq 0}$ *union with the holes.*

Representation of Lattice Points via Generating Functions

Given the parametric convex polytopes,
$P(b) = \{x | Ax = b,\ x \geq 0\}$,
GOAL: COUNT HOW MANY LATTICE POINTS are inside $P(b)$.



$\phi_A(b) = \#\{(x, y, z) | 3x + 5y + 17z = b,\ x \geq 0, y \geq 0, z \geq 0\}$

When $A = [3, 5, 17]$, a short formula for $\phi_A(n)$ would be a **generating function**

$$\sum_{n=0}^{\infty} \phi_A(n) t^n = \frac{1}{(1-t^{17})(1-t^5)(1-t^3)}.$$

From that, one can see that $\phi_A(100) = 25, \phi_A(1110) = 2471$, etc...

Theorem For a knapsack problem $A = [a_1, a_2, \dots, a_M]$, the generating function for $\phi_A(n)$ is

$$\sum_{n=0}^{\infty} \phi_A(n) t^n = \frac{1}{(1-t^{a_1})(1-t^{a_2})\dots(1-t^{a_M})}.$$

We can use it to count solutions for the coin problem!!

Computational Complexity

Theorem

Let $A \in \mathbb{Z}^{d \times n}$. Assuming that n and k are fixed, there is a polynomial time algorithm to compute a short sum of rational function $G(t)$ which efficiently represents the formal sum $\sum_{k-feasible} t^b$.
Here by k-feasible we mean that such precise description is possible for those b which are $=$ k-feasible, $\geq$ k-feasible, or $<$ k-feasible. Moreover, from the algebraic formula, one can perform the following tasks in polynomial time:

1. *Count the number of k-feasible vectors (if finite).*
2. *Extract the lexicographic-smallest b, k-feasible vector.*
3. *Find the k-feasible vector b that maximizes the dot product $c^T b$.*

- In 1993 A. Barvinok gave an algorithm for counting the lattice points in inside a polyhedron P in polynomial time when the dimension of P is a constant.
- The input of the algorithm is the inequality description of P, the output is a polynomial-size formula for the multivariate generating function of all lattice points in P, namely $f(P) = \sum_{a \in P \cap \mathbb{Z}^n} x^a$ where x^a is an abbreviation of $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$.
- A long polynomial with many many monomials is encoded as a much shorter sum of rational functions of the form

$$f(P) \quad = \quad \sum_{i \in I} \pm \frac{x^{u_i}}{(1 - x^{c_{1,i}})(1 - x^{c_{2,i}}) \dots (1 - x^{c_{n-d,i}})}. \quad (2)$$

- Barvinok and Woods developed a set of manipulation rules for using these short rational functions in **Boolean constructions** on various sets of lattice points.
- They also recover the lattice points inside the image a **linear projection** of a convex polytope.

- **Remark** From the results of Barvinok for fixed n, but not necessarily fixed k, one can decide whether a particular b is k-feasible in polynomial time, but more strongly

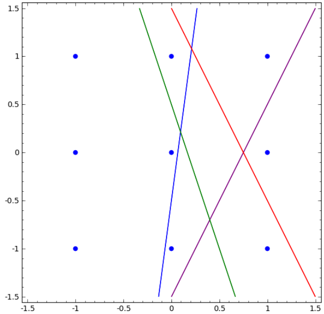
Corollary

Consider the knapsack problem $a^T x = b$ associated with $a = (a_1, \ldots, a_n)^T \in \mathbb{Z}^n_{>0}$ with $\gcd(a_1, \ldots, a_n) = 1$. For a fixed positive integer k and fixed n the k-Frobenius number can be computed in polynomial time.

- Identical results hold for the problem of the form $\{x : Ax \leq b, x \in \mathbb{Z}^n\}$.

A theorem of Doignon (reproved by Bell and Scarf)

- **Theorem** [Doignon 1973] Let A be a $d \times n$ matrix and b a vector of $\mathbb{R}^d$. If the problem $IP_A(\leq, b)$ is infeasible, then there is a subset S of the rows of A of cardinality no more than 2^n, with the property that the smaller integer program $IP_S(\leq, b)$ is also infeasible.



- This theorem has many applications, including Clarkson's probabilistic algorithm for integer linear programming.

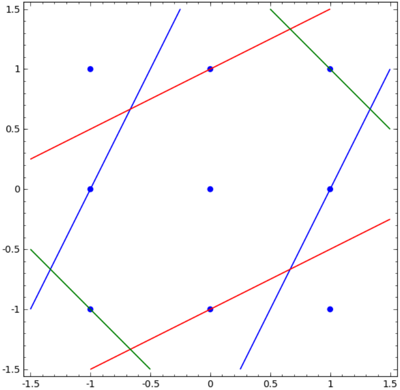
We proved a $=k$-feasibility version of Doignon's theorem:

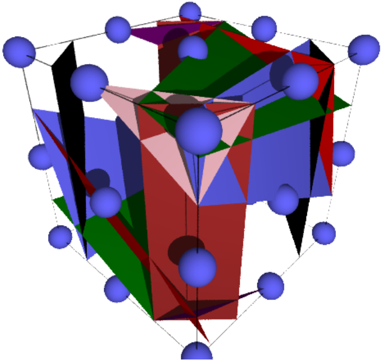
Theorem

Given n, k two non-negative integers there exists a universal constant $c(k,n)$, depending only on k and n, such that for any $d \times n$ integral matrix A, and d-vector b if $P_A(b)\{x : Ax \leq b\}$ has exactly k integral solutions, then there is a subset S of the rows of A of cardinality no more than $c(k,n)$, with the property that the smaller integer program $IP_S(\leq, b)$ has exactly the same k solutions as $P_A(b)$.

Our initial estimation of the constant $c(k,n)$ is $2^n 2^k$ but it appears to be loose!

Better values for dimension $c(2,1), c(3,1)$





Thank you!