Separation of singularities for holomorphic functions

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June, 2013

Problem

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- In the above problem one can consider the case of intersection of k domains
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- The problem can also be formulated for compact sets.

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- **Theorem 1.** Any function f holomorphic in $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ can be represented as $f_1 + f_2$, where f_i are holomorphic functions in $\mathcal{D}_i$, $i = 1, 2$.

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- Let $U(a, r)$ be the disk of radius r centered at $a \in \mathbb{C}$.
- A curve $\Gamma \subset \mathbb{C}$ is called Ahlfors-regular, if for any $a \in \mathbb{C}$ and any radius $r > 0$ the inequality $l(\Gamma \cap U(a, r)) \leq Cr$ holds, where l-is the length of the curve and the constant C does not depend on r and a.

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- **Theorem 2.** If the domain $\mathcal{D} = \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_k$ be bounded domains with Ahlfors-regular domains, then any function $f \in E^p(\mathcal{D})$, $1 < p < \infty$, can be represented as

$$f = f_1 + f_2 + \cdots + f_k,$$

where $f_i \in E^p(\mathcal{D}_i)$, $i = 1, 2, \ldots, k$.

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- Then

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  for every $j = 1, \ldots, k$.
- Each of the Cauchy integrals above belongs to the space $E^p(\mathcal{D}_j)$, $j = 1, \ldots, k$ (G.David, 1984).

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- **Theorem 2′.** *Let D be a finitely connected domain whose boundary consists of disjoint Jordan curves C_1, C_2,..., C_k. Let D be the intersection of the domains D_m, with boundary C_m, $m = 1, 2, .., k$. Then every $f \in H^p(D)$ can be represented as in Theorem 2, where $f_m \in H^p(D_m)$, $m = 1, 2, .., k$.*

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- D. Khavinson has kindly informed the author that this theorem can be generalized to the case of the intersection of the finite number of multiply connected domains.

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holomorphic in Ω, is not representable as a sum of holomorphic functions f_1 and f_2, defined in the domains Ω_1 and Ω_2, respectively.

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  holomorphic in Ω, is not representable as a sum of holomorphic functions f_1 and f_2, defined in the domains Ω_1 and Ω_2, respectively.
- Actually, if $f = f_1 + f_2$, where the function

$$f_j(z_1, z_2) = \sum_{n,m=0}^{\infty} a_{m,n}^{(j)} z_1^m z_2^n$$

  is holomorphic in Ω_j, $j = 1, 2$, then we choose $\epsilon > 0$ in a such a way that $(1 - \epsilon)(2 - \epsilon) > 1$.

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- Hence,

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- But the right hand-side of the last equality tends to 0 while $n \longrightarrow \infty$. Contradiction.

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- The following theorem is due to A. Weyl (1935).
Theorem 3. Any function, holomorphic in the polyhedron $\mathcal{D}$ can be written as a sum of functions f_J, holomorphic in larger domains $\mathcal{D}_J = \{z \in \Omega : \ |F_j(z)| < 1, \ j \in J\}$, where $|J| = n$. Here we use the notation $|(a_1, \ldots, a_k)| = a_1 + \cdots + a_k$.

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- **Theorem 4.** Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain satisfying $0 \in \mathcal{D}$. For $\epsilon > 0$, consider the domains $\Omega_1 = \{z \in \mathcal{D} : \Im z_1 > -\epsilon\}$ and $\Omega_2 = \{z \in \mathcal{D} : \Im z_1 < \epsilon\}$. If $S = \{z \in \mathcal{D} : \Im z_1 = 0\}$ is a real hypersurface in $\mathcal{D}$, then every function f holomorphic in the domain $\Omega_1 \cap \Omega_2$ can be expressed in S as a difference $f = f_1 - f_2$, where the function f_i is holomorphic in the domain Ω_i, $i = 1, 2$.

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- K.Oka in 1953 improved significantly the previous result as follows:
- **Theorem 5.** Every function f holomorphic in a neighborhood of the set S from Theorem 4 can be expressed as $f = f_+ - f_-$, where $f_\pm$ are functions holomorphic in a neighborhood of closures of the domains $\mathcal{D}_\pm = \{z \in \mathcal{D} : \ \pm\Im z_1 > 0\}$.

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- Let $E \subset \mathbb{C}^n$ containing the origin. We define its conjugate set to be

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- In what follows I will consider linearly convex domains which can be approximated from the inside by regular, linearly convex domains.

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- The domain $\mathcal{D} = \{z \in \mathbb{C}^n : \ \Phi(z, \bar{z}) < 0\}$ is called regular if its defining function Φ is $\mathcal{C}^{(2)}$, $grad\Phi(z, \bar{z}) \neq 0$, whenever $z \in \partial\mathcal{D}$ and there exists a ball $J_r = \{z \in \mathbb{C}^n : \ |z| < r\}$ so that on the set $\mathcal{D} \setminus J_r$ the expression $z_1\Phi'_{z_1} + \cdots + z_n\Phi'_{z_n}$ does not obtain negative values.

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- The domain $\mathcal{D} = \{z \in \mathbb{C}^n : \ \Phi(z, \bar{z}) < 0\}$ is called regular if its defining function Φ is $\mathcal{C}^{(2)}$, $grad\Phi(z, \bar{z}) \neq 0$, whenever $z \in \partial\mathcal{D}$ and there exists a ball $J_r = \{z \in \mathbb{C}^n : \ |z| < r\}$ so that on the set $\mathcal{D} \setminus J_r$ the expression $z_1\Phi'_{z_1} + \cdots + z_n\Phi'_{z_n}$ does not obtain negative values.
- For such domains, the following result is shown by A in 1967.
- **Theorem 6.** Any holomorphic function f in the domain $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$, that can be approximated from within by regular linearly convex domains, is written as in Theorem 2 for $k = 2$, where f_i is holomorphic in $\mathcal{D}_i$, $i = 1, 2$ if and only if the compactum of holomorphy $H(\widetilde{\mathcal{D}}_1 \cup \widetilde{\mathcal{D}}_2)$ for the union $\widetilde{\mathcal{D}}_1 \cup \widetilde{\mathcal{D}}_2$ satisfies $H(\widetilde{\mathcal{D}}_1 \cup \widetilde{\mathcal{D}}_2) = \widetilde{\mathcal{D}}$.

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- These are domains, whose intersection with any complex line is connected and simply connected, whenever it is not empty (they are also called $\mathbb{C}$-convex).
- **Theorem 7.** Let $\mathcal{D} = \mathcal{D}_1 \cdots \cap \ldots \mathcal{D}_k$ be strictly linearly convex domain. Any function f holomorphic in $\mathcal{D}$ is written as in Theorem 2, f_j are holomorphic functions in $\mathcal{D}_j$, $j = 1, 2 \ldots, k$ if and only if $H(\widetilde{\mathcal{D}}_1 \cup \cdots \cup \widetilde{\mathcal{D}}_k) = \widetilde{\mathcal{D}}$.

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- Similar results to Theorem 6 and Theorem 7 are valid for compact sets also. Besides the material quoted above there were many other results related to separation of singularities of holomorphic functions in many complex variables (see B. Mityagin and G. Henkin, Russian Mathematical Surveys, 1971, 26:4, 99-164, and the related literature therein).

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- In this direction we recall a result, due to G. Henkin (1997).
- **Theorem 8.** Let $\mathcal{D} = \{z \in \mathbb{C}^n : \varrho(z, \bar{z}) < 0\}$ be a domain, whose defining function ϱ is strictly prurisubharmonic function in a domain $\Omega \supseteq \overline{\mathcal{D}}$. let U_j, $j = 1, 2, 3, \ldots, k$ be open sets so that $\overline{\mathcal{D}} \subset U_1 \cup \cdots \cup U_k$. Then every bounded, holomorphic in $\mathcal{D}$ function f can be written as in Theorem 2, where every function f_j is holomorphic and bounded in a neighborhood of the set $\overline{\mathcal{D}} \setminus (\partial\mathcal{D} \cap U_j)$. Furthermore, if f is continuous in $\overline{\mathcal{D}}$, then all the functions f_j are continuous in $\overline{D}$.

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- Then, we say that $f \in H^p(\Omega)$, $p \geq 1$, if and only if

$$\limsup_{\epsilon \longrightarrow 0} \int\limits_{\partial\Omega} |f(\zeta - \epsilon\nu_\zeta)|^p d\sigma_\zeta < \infty,$$

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- The next result, A (2013), along with Theorem 2 are the main new results.
- **Theorem 9.** Consider the domain $\mathcal{D} = \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_k$, where all of the domains are strictly pseudoconvex with $\mathcal{C}^{(3)}$ boundary. Then every $f \in H^p(\mathcal{D})$, $1 < p < \infty$, is written as in Theorem 2, with $f_j \in H^p(\mathcal{D}_j)$, $j = 1, 2, \ldots, k$.

Proof

- If $\mathcal{D}$ is a strictly pseudoconvex domain of the type $\mathcal{D} = \{z \in \Omega : \ \varrho(z, \bar{z}) < 0\}$, where ϱ is a strictly prurisubharmonic function in the domain $\Omega \supset \overline{\mathcal{D}}$,

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- then for some neighborhood $U(\overline{\mathcal{D}})$ there exists a smooth function $\Phi(\zeta, z)$, $(\zeta, z) \in U(\overline{\mathcal{D}}) \times U(\overline{\mathcal{D}})$ such that Φ is holomorphic with respect to $z \in U(\overline{\mathcal{D}})$.

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- where $\Phi(\zeta, z) =< p(\zeta, z), \zeta - z >$ and $P = (p_1, \ldots, p_n)$ is a smooth vector function in $U(\overline{\mathcal{D}}) \times U(\overline{\mathcal{D}})$, holomorphic with respect to $z \in U(\overline{\mathcal{D}})$.

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- where $\omega(\zeta) = d\zeta_1 \wedge \ldots d\zeta_n$,

$$\omega'(p(\zeta,z)) = \sum_{j=1}^{n} (-1)^{j-1} p_j dp_1 \wedge \cdots \wedge dp_{j-1} \wedge dp_{j+1} \wedge \cdots \wedge dp_n.$$

Proof

- We now decompose the boundary $\partial\mathcal{D}$ into k components $M_j = \partial\mathcal{D}_j \cap \partial\mathcal{D}$, $j = 1, 2, ..., k$, corresponding to the parts in $\mathcal{D} = \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_k$.

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- Then the last integral is represented as a sum of integrals of Cauchy-Leray type over the boundary $\partial\mathcal{D}_j$ of functions

$$F_j(t) = \begin{cases} f(t), \ if \ t \in M_j \\ 0, \ t \in \partial\mathcal{D}_j \setminus M_j, \end{cases}$$

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  for every $j = 1, \ldots, k$.
- Each of these integrals belongs to the class $H^p(\mathcal{D}_j)$, $j = 1, 2, \ldots, k$. This follows from results found by N. Kerzman and E.M. Stein in 1978, where they were stated for domains with $\mathcal{C}^\infty$ boundaries, but in the proofs the facts used were that the smoothness of the boundary was up to degree 3.

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- If one considers the same problem for a class of holomorphic functions with reasonable behavior (in some sense) near the boundary $\partial\mathcal{D}$,
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- It is shown in the following example that Theorem 9 is false for domains from Example 1.

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- then $f \in H^2(U_{r,\rho})$ if and only if
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$$\sum_{m,n=0}^{\infty} |a_{m,n}|^2 r^{2m} \rho^{2n} < \infty.$$
- Let $f = f_1 + f_2$, $f \in H^2(U_{1,1})$, $f_1 \in H^2(U_{2,1})$, $f_2 \in H^2(U_{1,2})$. Consider particular
$$f(z) = \sum_{m,n=0}^{\infty} \frac{z_1^m z_2^n}{mn}, \quad f_j(z) = \sum_{m,n=0}^{\infty} a_{m,n}^{(j)} z_1^m z_2^n, \ j = 1, 2.$$

- Then, on the diagonal $m = n$ one has that

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- Then, on the diagonal $m = n$ one has that

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- which means that there exists a constant $C_1 > 0$ so that $|a_{m,m}^{(1)}| 2^m < C_1$.
- Similarly, there exists a constant $C_2 > 0$ so that $|a_{m,m}^{(2)}| 2^m < C_2$.

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- which means that there exists a constant $C_1 > 0$ so that $|a^{(1)}_{m,m}| 2^m < C_1$.
- Similarly, there exists a constant $C_2 > 0$ so that $|a^{(2)}_{m,m}| 2^m < C_2$.
- Thus, for all m one has that

$$\frac{1}{m^2} < |a^{(1)}_{m,m}| + |a^{(2)}_{m,m}| < \frac{C_1 + C_2}{2^m},$$

contradiction.