# The realizability problem as an infinite dimensional moment problem


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# Outline

# Motivation: analysis of complex systems

Nervous system in which neural spikes travel

Fluid composed of molecules

Plants growing in a certain region

Dealing with MANY-BODY SYSTEMS

Galaxy composed of stars

STRATEGY:

- focus on few characteristics of the system
- predict the selected characteristics

OUTPUT: Putative characteristics of the system

**REALIZABILITY PROBLEM**

Does there exist a real state of the original system which realizes the putative characteristics predicted?

# Framework: Configuration space

- **Point configuration** $\to \gamma \subset \mathbb{R}^d \ : \ |\gamma \cap \Lambda| < \infty, \forall \Lambda \subset \mathbb{R}^d$ compact

$$\boxed{\gamma(d\mathbf{r}) = \sum_{i \in I} \delta_{\mathbf{x}_i}(d\mathbf{r})}$$

with $\mathbf{x}_i \in \mathbb{R}^d$, either $I \subset \mathbb{N}$ is finite or $I = \mathbb{N}$ and $(x_i)_{i \in I}$ has no accumulation points.

- $\Gamma(\mathbb{R}^d) :=$ set of all possible point configurations on $\mathbb{R}^d$

$$\Gamma(\mathbb{R}^d) \subset \mathcal{R}(\mathbb{R}^d)$$

where $\mathcal{R}(\mathbb{R}^d)$ is the set of all Radon measures.

- **Point process** $\mu :=$ a probability measure on $\Gamma(\mathbb{R}^d)$.

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# Point configurations and complex systems

Points $\mathbf{x}_i$ of a configuration $\gamma \leftrightarrow$ Positions of the entities of the system.

For example, $x_i$ may represent the location of:

- molecules in a fluid
- stars in galaxies
- trains of neural spikes
- trees of a plant population

## References

- Heterogenous materials: F. H. Stillinger, S. Torquato
- Liquids: J. K. Percus
- Quantum chemistry: P. O. Lödwin, A. J. Coleman, J. K. Percus
- Spatial ecology: B. Bolker, R. Law, H. Metz

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Low-order correlation functions

- contain many properties of physical interest
- are the most accessible to experiments

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- **one-particle density** $\rightarrow \rho_\mu^{(1)}(d\mathbf{r}_1) := \mathbb{E}_\mu(\gamma(d\mathbf{r}_1))$,
  i.e. expected number of particles in the infinitesimal volume $d\mathbf{r}_1$.

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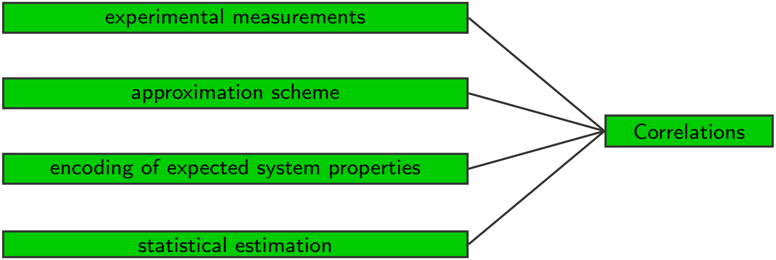
- **pair density** $\rightarrow \rho_\mu^{(2)}(d\mathbf{r}_1, d\mathbf{r}_2) := \mathbb{E}_\mu(\gamma^{\odot 2}(d\mathbf{r}_1, d\mathbf{r}_2))$,
  i.e. expected number of pairs of particles with one particle in the infinitesimal volume $d\mathbf{r}_1$ and the other in $d\mathbf{r}_2$.

# Realizability problem

Given a point process $\mu$ the correlation functions can be calculated in principle, even if this may be analytically challenging and often impossible in practice.



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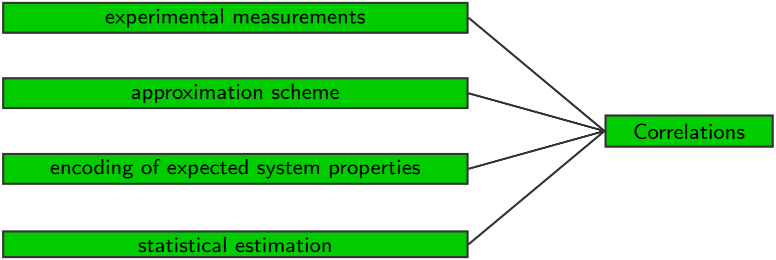


**Realizability Problem for correlations**

Given two putative correlations $\rho_1(\mathbf{r}_1)$ and $\rho_2(\mathbf{r}_1, \mathbf{r}_2)$, does there exist any point process $\mu$ such that $\rho_1$ and $\rho_2$ are the correlation functions of $\mu$?

# Powers and factorial powers of configurations

For any point configuration $\gamma(d\mathbf{r}) = \sum_{i\in I} \delta_{x_i}(d\mathbf{r})$ we define

- ***n*−th power** of $\gamma$

$$\boxed{\gamma^{\otimes n}(d\mathbf{r}_1, \dots, d\mathbf{r}_n) := \gamma(d\mathbf{r}_1)\cdots\gamma(d\mathbf{r}_n) = \sum_{i_1,\dots,i_n} \prod_{h=1}^{n} \delta_{x_{i_h}}(d\mathbf{r}_h),}$$

- **Factorial *n*−th power** of $\gamma$

$$\boxed{\gamma^{\odot n}(d\mathbf{r}_1, \dots, d\mathbf{r}_n) := \sum_{i_1\neq\dots\neq i_n} \prod_{h=1}^{n} \delta_{x_{i_h}}(d\mathbf{r}_h).}$$

Note that it is called factorial power of $\gamma$ because for any measurable $A$

$$\gamma^{\odot n}(A \times \cdots \times A) = \gamma(A)!$$

# Correlation functions and moment functions

$\rho_\mu^{(n)} := \mathbb{E}_\mu(\gamma^{\odot n}) \rightsquigarrow n-$th correlation function of $\mu$

$m_\mu^{(n)} := \mathbb{E}_\mu(\gamma^{\otimes n}) \rightsquigarrow n-$th moment function of $\mu$

Then from the relation between $\gamma^{\otimes n}$ and $\gamma^{\odot n}$ it follows that:

- I. order: $\rho_\mu^{(1)}(d\mathbf{r}) = m_\mu^{(1)}(d\mathbf{r})$.
- II. order:
  $\rho_\mu^{(2)}(d\mathbf{r}_1, d\mathbf{r}_2) = m_\mu^{(2)}(d\mathbf{r}_1, d\mathbf{r}_2) - \delta_{\mathbf{r}_1}(d\mathbf{r}_2) m_\mu^{(1)}(d\mathbf{r}_2)$.

**WARNING:**
$\rho_\mu^{(2)}(d\mathbf{r}_1, d\mathbf{r}_2)$ is NOT the variance of $\mu$!

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**Realizability Problem for correlations** $\longleftrightarrow$ **Realizability Problem for moment functions**

# Realizability problem on $\Gamma(\mathbb{R}^d)$ for moment functions

For any $\gamma \in \Gamma(\mathbb{R}^d)$, $n \in \mathbb{N}_0$ and $f^{(n)} \in \mathcal{C}_c(\mathbb{R}^{dn})$, we define

$$\langle f^{(n)}, \gamma^{\otimes n} \rangle \quad := \quad \int_{\mathbb{R}^{dn}} f^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) \gamma^{\otimes n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \sum_{i_1, \ldots, i_n} f^{(n)}(x_{i_1}, \ldots, x_{i_n}).$$

For convention, $\langle f^{(0)}, \gamma^{\otimes 0} \rangle := f^{(0)}$ with $f^{(0)} \in \mathbb{R}$.

**RP on $\mathcal{S} \subseteq \Gamma(\mathbb{R}^d)$ for moment functions**

Given a sequence $m = (m^{(n)})_{n=0}^N$ of symmetric Radon measures on $\mathbb{R}^{dn}$, find a point process $\mu$ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$\langle f^{(n)}, m^{(n)} \rangle = \int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\otimes n} \rangle \mu(d\gamma), \quad \forall f^{(n)} \in \mathcal{C}_c(\mathbb{R}^{dn}),$$

i.e. $m^{(n)}$ is the $n-th$ moment function of $\mu$.

# Interpretation of RP as infinite dimensional MP

Let $N \in \mathbb{N} \cup \{\infty\}$.

**Moment problem on $K \subseteq \mathbb{R}$**

Given a sequence $(m_n)_{n=0}^N$ with $m_n \in \mathbb{R}$, find a finite nonnegative Borel measure $\mu$ concentrated on $K$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} .$$

**Realizability Problem on $\mathcal{S} \subseteq \Gamma(\mathbb{R}^d)$**

Given a sequence $(m^{(n)})_{n=0}^N$ with $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ symmetric, find a finite nonnegative measure $\mu$ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

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# Interpretation of RP as infinite dimensional MP

Let $N \in \mathbb{N} \cup \{\infty\}$. $N = \infty \rightsquigarrow$ Full MP/RP $N \in \mathbb{N} \rightsquigarrow$ Truncated MP/RP

## Moment problem on $K \subseteq \mathbb{R}$

Given a sequence $(m_n)_{n=0}^N$ with $m_n \in \mathbb{R}$, find a finite nonnegative Borel measure $\mu$ concentrated on $K$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} .$$

## Realizability Problem on $\mathcal{S} \subseteq \mathcal{R}(\mathbb{R}^d)$

Given a sequence $(m^{(n)})_{n=0}^N$ with $m^{(n)} \in \mathcal{R}(\mathbb{R}^d)$ symmetric, find a finite nonnegative measure $\mu$ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$\langle f^{(n)}, m^{(n)} \rangle = \underbrace{\int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\otimes n} \rangle \mu(d\gamma)}_{n\text{-th moment of } \mu}, \quad \forall f^{(n)} \in \mathcal{C}_c(\mathbb{R}^{dn}).$$

# Projections of RP

Let us consider $\mathcal{S} \subseteq \mathcal{R}(\mathbb{R}^d)$. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we define:

$$\begin{array}{rcl} N_\Lambda : (\mathcal{S}, \mathcal{B}(\mathcal{S})) & \to & (K_\mathcal{S}, \mathcal{B}(K_\mathcal{S})) \\ \gamma & \mapsto & N_\Lambda(\gamma) := \gamma(\Lambda) \end{array}$$

## Definition (Associated moments)

$$m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn}) \text{ and } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \longrightarrow m_{n,\Lambda} := m^{(n)}(\Lambda^n) \in \mathbb{R}.$$

## Theorem

*If $m = (m^{(n)})_{n=0}^N$ is realized by a measure $\mu$ on $\mathcal{S}$, then for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the sequence $(m_{n,\Lambda})_{n=0}^N$ is realized by $\mu_{N_\Lambda}$ on $K_\mathcal{S}$ where $K_\mathcal{S} = N_\Lambda(\mathcal{S})$.*

For instance:

$$K_\mathcal{S} = \begin{cases} \mathbb{R}^+ & \text{if } \mathcal{S} = \mathcal{R}(\mathbb{R}^d) \\ \mathbb{N}_0 & \text{if } \mathcal{S} = \Gamma(\mathbb{R}^d) \\ [0,1] & \text{if } \mathcal{S} = \mathcal{P}(\mathbb{R}^d) \end{cases}$$

# Riesz's Functional Non-negativity

## Riesz's Functional

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

$$\begin{array}{llcl} L_m : & \mathbb{R}[x] & \to & \mathbb{R} \\ & p(x) := \sum_{n=0}^{N} a_n x^n & \mapsto & L_m(p) := \sum_{n=0}^{N} a_n m_n. \end{array}$$

**Note:**

If $m$ is realized by a non-negative measure $\mu$ on $K$, then

$$L_m(p) = \sum_{n=0}^{N} a_n m_n = \sum_{n=0}^{N} a_n \int_K x^n \mu(dx) = \int_K p(x)\mu(dx).$$

Hence

$$(m \text{ is realizable on } K) \Rightarrow (\forall p \geq 0 \text{ on } K,\ L_m(p) \geq 0)$$

# Riesz's Functional Non-negativity

**Theorem [Riesz, 1923 ($d = 1$) - Haviland, 1936 ($d \geq 2$)]**

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$ and $K \subseteq \mathbb{R}^d$ closed.
$m$ is realized by a nonnegative measure $\mu$ on K iff $L_m$ is $K$-nonnegative, i.e.

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**Problem: Limited Applicability!**

Classification of all polynomials nonnegative on a general closed set $K \subseteq \mathbb{R}^d$

$$\rightsquigarrow$$

**impractical conditions**

# Classification of nonnegative polynomials and sum of squares

When $K = \mathbb{R}$ then the Riesz-Haviland conditions are actually "more concrete".

**Theorem. (Hilbert, 1888)**

Every polynomial nonnegative on $\mathbb{R}$ can be expressed as sum of squares of polynomials, i.e.

$$\forall\, p \in \mathbb{R}[x] : \ p \geq 0 \text{ on } \mathbb{R}\,,\ \exists p_1, p_2 \in \mathbb{R}[x] \ : \ p = p_1^2 + p_2^2.$$

Then, for any $p \in \mathbb{R}[x]$ we have

$$L_m(p) = L_m(p_1^2) + L_m(p_2^2),$$

and appling the Riesz-Haviland theorem we get the following:

**Theorem. (Hamburger, 1921)**

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

$$\Big(m \text{ is realizable on } K = \mathbb{R}\Big) \iff \Big(\forall p \in \mathbb{R}[x],\ L_m(p^2) \geq 0\Big)$$

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$$\Big(m \text{ is realizable on } K = \mathbb{R}\Big) \iff \Big(\forall p \in \mathbb{R}[x],\; L_m(p^2) \geq 0\Big)$$

**Theorem. (Carleman, 1926)**

$$\left(m \text{ realized by } \mu \text{ on } \mathbb{R} \text{ s.t. } \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty\right) \Rightarrow \Big(\mu \text{ is the unique realizing measure}\Big)$$

# Hankel Matrices p.s.d.

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

## Hankel matrix

Let $N \in \mathbb{N}$. The Hankel matrix of order $N$ associated to $m$ is given by

$$H_N(m) := \left(m_{i+j}\right)_{i,j=0}^{N} = \begin{pmatrix} m_0 & m_1 & m_2 & \dots & m_N \\ m_1 & m_2 & m_3 & \dots & m_{N+1} \\ m_2 & m_3 & m_4 & \dots & m_{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_N & m_{N+1} & m_{N+2} & \dots & m_{2N} \end{pmatrix}$$

## Positive semi-definiteness (p.s.d.)

The sequence $m$ is said to be positive semi-definite iff

$$\forall\, N \in \mathbb{N}, \quad H_N(m) \succeq 0,$$

or equivalently iff

$$\forall p \in \mathbb{R}[x], \quad L_m(p^2) \geq 0.$$

# Classical results in one dimension: P.s.d. type conditions

- **Hamburger**, 1921

$$\Big(m \text{ realizable on } K = \mathbb{R}\Big) \iff \Big(\forall p \in \mathbb{R}[x],\ L_m(1p^2) \geq 0\Big)$$

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- **Stieltjes**, 1886–1895

$$\Big(m \text{ realizable on } K=\mathbb{R}^+\Big) \iff \Big(\forall p\in\mathbb{R}[x],\ L_m(1p^2)\geq 0,\ L_m(xp^2)\geq 0\Big)$$

$$\iff \Big((m_n)_{n=0}^{\infty} \text{ and } (m_{n+1})_{n=0}^{\infty} \text{ p.s.d.}\Big)$$

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- **Hausdorff**, 1920

$$\left(m \text{ is realizable on } K = [0,1]\right) \iff \left(\forall p, L_m(1p^2) \geq 0,\ L_m(xp^2) \geq 0,\ L_m((1-x)p^2) \geq 0\right)$$

$$\iff \left((m_n)_{n=0}^{\infty},\ (m_{n+1})_{n=0}^{\infty},\ (m_n - m_{n+1})_{n=0}^{\infty} \text{ p.s.d.}\right)$$

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$$\iff \Big((m_n)_{n=0}^{\infty}, (m_{n+1})_{n=0}^{\infty}, (m_n - m_{n+1})_{n=0}^{\infty} \text{ p.s.d.}\Big)$$

Instead of checking $L_m(p) \geq 0$ for all $p \geq 0$ on $K$, we only need check that $L_m$ is non-negative on a finite set of "test polynomials" multiplied the squares of polynomials.

# MP on one dimensional basic semi-algebraic sets

**Basic closed semi-algebraic set of $\mathbb{R}$**

$$K = \bigcap_{j=1}^{m} \{x \in \mathbb{R} | P_j(x) \geq 0\}, \quad \text{where } P_j \in \mathbb{R}[x] \text{ for } j = 1, \ldots, m.$$

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**Theorem. (Berg-Maserick, 1982)**

$K \subset \mathbb{R}$ basic semi-algebraic COMPACT generated by $P_1, \ldots, P_m$

$$\left(\exists! \mu \text{ realizing } m \text{ on } K\right) \Leftrightarrow \left(\forall p \in \mathbb{R}[x],\ \forall j \in \{1, \ldots, m\},\ \begin{array}{l} L_m(p^2) \geq 0 \\ L_m(P_j p^2) \geq 0 \end{array}\right)$$

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**Uniqueness**

$\Big(\text{K COMPACT}\Big) + \Big(m \text{ realizable on } K\Big) \Rightarrow m$ satisfies Carleman's condition.

# Multidimensional MP

$\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

## Moment problem on $K \subseteq \mathbb{R}^d$

Given a sequence $m = (m_\alpha)_\alpha$ with $m_\alpha \in \mathbb{R}$, find a finite nonnegative Borel measure $\mu$ concentrated on $K$ s.t. for any $\alpha \in \mathbb{N}_0^d$ we have

$$m_\alpha = \int_K \mathbf{x}^\alpha \mu(d\mathbf{x}),$$

i.e.

$$m_{(\alpha_1,\dots,\alpha_d)} = \int_K x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mu(dx_1, \dots, dx_d).$$

For $d \geq 2$, there exist non-negative polynomials on $\mathbb{R}^d$ which are not sums of squares!

# Multidimensional MP

$\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

**Moment problem on $K \subseteq \mathbb{R}^d$**

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For $d \geq 2$, there exist non-negative polynomials on $\mathbb{R}^d$ which are not sums of squares!

**Theorem. (Nussbaum, 1965)**

Let $m = (m_\alpha)_\alpha$ with $m_\alpha \in \mathbb{R}$ and for $h = 1, \dots, d$ $\sum_{k=1}^{\infty} \left(m_{(0,\dots,0,\underbrace{2k}_{h-th},0,\dots,0)}\right)^{-\frac{1}{2k}} = \infty$.

$$\left(\exists! \, \mu \text{ realizing } m \text{ on } \mathbb{R}^d\right) \iff \left(\forall p \in \mathbb{R}[\mathbf{x}], \; L_m(p^2) \geq 0\right)$$

# MP on finite dimensional basic semi-algebraic sets

**Basic closed semi-algebraic set of $\mathbb{R}^d$**

$$K = \bigcap_{j=1}^{m} \{\mathbf{x} \in \mathbb{R}^d | P_j(\mathbf{x}) \geq 0\}, \quad \text{where } P_j \in \mathbb{R}[\mathbf{x}].$$

To this geometric object let us associate two algebraic objects.
For every $J \subseteq \{1, \ldots, m\}$ let $P_J := \prod_{k \in J} P_k$ and $P_0 \equiv P_\emptyset \equiv 1$.

- **Preordering associated to** $K$

$$\mathcal{P}(K) := \left\{ \sum_{J \subseteq \{1,\ldots,m\}} \sigma_J P_J \; : \; \sigma_J \in \Sigma[\mathbf{x}] \right\}.$$

- **Quadratic module associated to** $K$

$$\mathcal{Q}(K) := \left\{ \sum_{j=0}^{m} \sigma_j P_j \; : \; \sigma_j \in \Sigma[\mathbf{x}] \right\}.$$

$\mathcal{Q}(K)$ is **Archimedean** if there exists $N \in \mathbb{N}$ s.t. $N - \|\mathbf{x}\|^2 \in \mathcal{Q}(K)$.

# MP on COMPACT basic semi-algebraic sets of $\mathbb{R}^d$

The first result on the multidimensional moment problem which covers a truly general class of sets $K$, rather than just specific sets is due to Schmüdgen, who proved the following remarkable theorem.

**Theorem. (Schmüdgen, 1991)**

$K \subset \mathbb{R}^d$ basic semi-algebraic COMPACT

$$\left(\exists! \mu \text{ realizing } m \text{ on } K\right) \Leftrightarrow \left(\forall p \in \mathbb{R}[x],\ \forall J \subseteq \{1,\dots,m\},\ \begin{array}{l} L_m(p^2) \geq 0 \\ L_m(p^2 \prod_{i \in J} P_i) \geq 0 \end{array}\right)$$
$$\Leftrightarrow (\forall f \in \mathcal{P}(K),\ L_m(f) \geq 0)$$

Schmüdgen's theorem was soon refined by Putinar for Archimedean quadratic modules.

**Theorem. (Putinar, 1993)**

$K \subset \mathbb{R}^d$ basic semi-algebraic COMPACT$+\ \mathcal{Q}(K)$ Archimedian

$$\left(\exists! \mu \text{ realizing } m \text{ on } K\right) \Leftrightarrow \left(\forall p \in \mathbb{R}[x],\ \forall i \in \{1,\dots,m\},\ \begin{array}{l} L_m(p^2) \geq 0 \\ L_m(P_i p^2) \geq 0 \end{array}\right)$$
$$\Leftrightarrow (\forall f \in \mathcal{Q}(K),\ L_m(f) \geq 0)$$

Berg-Maserick's theorem is a particular case of Putinar's theorem for $d = 1$.

# Overview MP on finite dimensional basic semi-algebraic sets

$K$ basic semi-alg COMPACT:

- **Schmüdgen**, 1991:
  $\left(\exists! \mu \text{ realizing } m \text{ on } K\right) \Leftrightarrow (\forall f \in \mathcal{P}(K),\ L_m(f) \geq 0.)$
- **Putinar**, 1993: Assume $\mathcal{Q}(K)$ is Archimedean.
  $\left(\exists! \mu \text{ realizing } m \text{ on } K\right) \Leftrightarrow (\forall f \in \mathcal{Q}(K),\ L_m(f) \geq 0.)$

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$K$ basic closed semi-alg. NOT COMPACT:

KMP solvable via $\mathcal{P}(K)$ or $\mathcal{Q}(K)$ ONLY for specific classes of semi-algebraic sets

- **Powers, Scheiderer, 2001**,
- **Schmüdgen, 2003**
- **Kuhlman, Marshall, 2002**
- **Scheiderer, 2009**

# Overview MP on finite dimensional basic semi-algebraic sets

$K$ basic semi-alg COMPACT:

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Idea:

$L_m$ continuous w.r.t. a topology which is not the finest locally convex top. on $\mathbb{R}[\mathbf{x}]$.

- **Ghasemi, Kuhlmann, Samei, 2013**
- **Lasserre, 2013**

# MP on finite dimensional basic closed semi-algebraic sets

**Theorem.** (Lasserre, 2013)

$K \subseteq \mathbb{R}^d$ basic closed semi-alg. $+$ $L_m$ is $\ell_w-$continuous

$(\exists! \mu$ realizing $m$ on $K) \iff (\forall f \in \mathcal{Q}(K),\ L_m(f) \geq 0.)$

$$\left(L_m \text{ is } \ell_w - \text{continuous}\right) \Leftrightarrow \left(\exists R > 0 : \ \sup_{\alpha \in \mathbb{N}^d} \frac{|m_\alpha|}{w_\alpha} \leq R \text{ with } w_\alpha := (2\lceil |\alpha|/2 \rceil)!\right)$$
$$\Rightarrow \left(\exists R > 0 \ : \ h = 1, \ldots, d, \forall k \in \mathbb{N},\ |m_{(0,\ldots,0,\underbrace{2k}_{h-th},0,\ldots,0)}| \leq (2k)!R\right)$$

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$$(\exists! \mu \text{ realizing } m \text{ on } K) \iff \left( \forall p \in \mathbb{R}[x],\ \forall j \in J,\ \begin{array}{l} L_m(p^2) \geq 0 \\ L_m(P_j p^2) \geq 0 \end{array} \right)$$

# MP on finite dimensional basic closed semi-algebraic sets

**Theorem. (Lasserre, 2013)**

$K \subseteq \mathbb{R}^d$ basic closed semi-alg. + $L_m$ is $\ell_w-$continuous

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The bound in blue implies the multivariate Carleman's condition.

# Riesz's Functional Non-negativity in infinite dimensions

Let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^d)$.

**Riesz's Functional**

$$\begin{array}{rlcl} L_m : & \mathcal{P}_{\mathcal{C}_c}(\mathcal{R}(\mathbb{R}^d)) & \to & \mathbb{R} \\ & p(\gamma) := \sum_{n=0}^{N} \langle a^{(n)}, \gamma^{\otimes n} \rangle & \mapsto & L_m(p) := \sum_{n=0}^{N} \langle a^{(n)}, m^{(n)} \rangle. \end{array}$$

**Theorem (Lenard, 1975)**

Let $\mathcal{S} \subseteq \mathcal{R}(\mathbb{R}^d)$ closed w.r.t. the vague topology on $\mathcal{R}(\mathbb{R}^d)$

$$(m \text{ is realizable on } \mathcal{S}) \iff (\forall p \geq 0 \text{ on } \mathcal{S},\ L_m(p) \geq 0)$$

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**Problem: Limited Applicability!**

Characterization of nonnegative polynomials on $\mathcal{S} \subseteq \mathcal{R}(\mathbb{R}^d)$.

# Positive semidefinite approach in infinite dimensions

## MP for correlations on configuration spaces

- Berezansky, Kondratiev, Kuna, Lytvynov, 1999
- Kondratiev, Kuna, 2002
- Kondratiev, Kuna, Oliveira, 2006
- Lytvynov, Lin Mei, 2007

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**MP for moment functions on nuclear spaces**: Berezansky-Kondratiev, 1988

- $\Omega = \operatorname{proj}\lim_{k\in K} H_k$ and nuclear, where $(H_k)_{k\in K}$ is a family of Hilbert spaces.
- $\Omega' =$ dual of $\Omega$ w.r.t. the topology $\tau_{proj}$
- $n$-th tensor power $\rightsquigarrow \Omega^{\otimes n} := \operatorname{proj}\lim_{k\in K} H_k^{\otimes n}$

**MP on nuclear spaces**

Let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \left(\Omega^{\otimes n}\right)'$ is symmetric.
Does there exist $\mu$ concentrated on $\Omega'$ s.t. for any $f^{(n)} \in \Omega^{\otimes n}$

$$\left\langle f^{(n)}, m^{(n)} \right\rangle = \int_{\Omega'} \left\langle f^{(n)}, \gamma^{\otimes n} \right\rangle \mu(d\gamma)?$$

# Positive semidefinite conditions in infinite dimensions

## MP on nuclear space

Let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \left(\Omega^{\otimes n}\right)'$ is symmetric.

**Positive semidefinite sequence**

$m$ is *positive semidefinite* iff $\forall p \in \mathcal{P}_\Omega(\Omega'),\ L_m(p^2) \geq 0$.

**Determining sequence**

$m$ is *determining* if $\exists\, k(m) \in K$ s.t. for any $n \in \mathbb{N}$ and for any $f_1, \dots, f_{2n} \in \Omega$ we have

$$\left|\left\langle f_1 \otimes \cdots \otimes f_{2n}, m^{(2n)} \right\rangle\right| \leq \tilde{m}_n^2 \prod_{l=1}^{2n} \|f_l\|_{H_{k(m)}},$$

where $\tilde{m}_n \in \mathbb{R}^+$ are such that the class $C\{\tilde{m}_n\}$ is quasi-analytical.

**Theorem. (Berezansky-Kondratiev, 1988)**

Assume $m$ is determining.
($m$ positive semidefinite) $\iff$ ($\exists! \mu$ non-negative on $\Omega'$ realizing $m$)

# Our idea

Berezansky-Kondratiev result applied to $\mathscr{D}'(\mathbb{R}^d)$

Ideas coming from the results on finite dimensional semi-algebraic sets

P.s.d. type conditions for realizability on $\mathcal{S} \subseteq \mathscr{D}'(\mathbb{R}^d)$ basic semi-algebraic

# Our result

- $\Omega := \mathcal{C}_c^\infty(\mathbb{R}^d) = \underset{k\in K}{\text{proj lim}}\, W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ nuclear, [Berezansky, 1986].
- $\Omega' = \mathscr{D}'(\mathbb{R}^d)$ i.e. the dual of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ w.r.t. the projective limit topology.
- $\mathcal{S} = \bigcap_{j\in J} \left\{\gamma \in \mathscr{D}'(\mathbb{R}^d) \mid P_j(\gamma) \geq 0\right\}$, where $J$ is **not necessarily countable**
- $\mathcal{Q}(\mathcal{S}) := \bigcup_{\substack{J_0\subset J\\ |J_0|<\infty}} \left\{\sum_{j\in J_0} Q_j P_j \,:\, Q_j \in \Sigma(\mathscr{D}'(\mathbb{R}^d))\right\}.$

  W.l.o.g. we assume that $0 \in J$ and we define $P_0(\mathbf{r}) = 1$ for all $\mathbf{r} \in \mathbb{R}^d$.

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  W.l.o.g. we assume that $0 \in J$ and we define $P_0(\mathbf{r}) = 1$ for all $\mathbf{r} \in \mathbb{R}^d$.

**Theorem. (I., Kuna, Rota)**

Let $m = (m^{(n)})_{n=0}^\infty$ be a sequence such that $m^{(n)} \in \mathscr{D}'(\mathbb{R}^{dn})$ is a symmetric function of its $n$ variables. Assume that $m$ is determining.

$$\begin{aligned}(\exists!\mu \text{ realizing } m \text{ on } \mathcal{S}) \quad &\Leftrightarrow (\forall f \in \mathcal{Q}(\mathcal{S}),\ L_m(f) \geq 0)\\ &\Leftrightarrow \left(\forall p \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'(\mathbb{R}^d)\right), \forall j \in J,\ \begin{array}{l} L_m(p^2) \geq 0 \\ L_m(P_j p^2) \geq 0 \end{array}\right)\end{aligned}$$

Proof.

# Realizability on Radon measures

**$\mathcal{R}(\mathbb{R}^d)$ as semi-algebraic set**

$$\mathcal{R}(\mathbb{R}^d) = \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \{\gamma \in \mathscr{D}'(\mathbb{R}^d) | \langle \varphi, \gamma \rangle \geq 0\}$$

**Conditions for realizability of a sequence of Radon measures $m$ on $\mathcal{R}(\mathbb{R}^d)$**

For all $h^{(i)} \in \mathcal{C}_c^{\infty}(\mathbb{R}^{id})$ and for all $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$

$$\sum_{i,j} \int_{\mathbb{R}^{id}} \int_{\mathbb{R}^{jd}} h^{(i)}(x) h^{(j)}(y) m^{(i+j)}(dx, dy) \geq 0$$

and

$$\sum_{i,j} \int_{\mathbb{R}^{id}} \int_{\mathbb{R}^{jd}} \int_{\mathbb{R}^{d}} h^{(i)}(x) h^{(j)}(y) \varphi(z) m^{(i+j+1)}(dx, dy, dz) \geq 0.$$

# Realizability on configuration spaces

## Multiple configuration space as semi-algebraic set

$$\ddot{\Gamma}(\mathbb{R}^d) = \bigcap_{k\in\mathbb{N}} \bigcap_{\varphi\in\mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{\gamma \in \mathscr{D}'(\mathbb{R}^d) : \langle \varphi^{\otimes k}, \gamma^{\odot k}\rangle \geq 0\right\}.$$

Conditions involve infinitely many polynomial of arbitrarily large degree!

$$\Downarrow$$

Realizability on SIMPLE point configuration space

$$\Gamma(\mathbb{R}^d) := \left\{\gamma \in \Gamma(\ddot{\mathbb{R}}^d) \ : |\gamma \cap \{\mathbf{x}\}| \in \{0,1\}, \forall\, \mathbf{x} \in \mathbb{R}^d\right\}$$

## Theorem

*m determining.*

$$(\exists!\mu \text{ realizing } m \text{ on } \Gamma) \Leftrightarrow \begin{pmatrix} L_m(p^2) \geq 0, \\ L_m(\langle\varphi,\eta\rangle p^2) \geq 0,\ L_m(\langle\varphi^{\otimes 2},\eta^{\odot 2}\rangle p^2) \geq 0 \\ m^{(2)}(diag(\Lambda\times\Lambda)) = m^{(1)}(\Lambda),\ \ \forall\,\Lambda\, compact \end{pmatrix}$$

*Thank you for your attention*

# Our result

**TRICK:** Generalized shifted sequence

Given $m = (m^{(n)})_{n=0}^{\infty}$ and $Q(\gamma) := \sum_{i=0}^{N} \langle q^{(i)}, \gamma^{\otimes i} \rangle$,define

$$\boxed{\langle f^{(n)}, ({}_{Q}m)^{(n)} \rangle := \sum_{i=0}^{N} \langle q^{(i)} \otimes f^{(n)}, m^{(n+i)} \rangle.}$$

Then

$$L_{{}_{Q}m}(p^2) = L_m(Qp^2)$$

and so

$$(\forall p,\ \forall j,\ \ L_m(P_j p^2) \geq 0) \iff (\forall j,\ \ {}_{P_j}m \text{ p.s.d.})$$

**Theorem. (I., Kuna, Rota)**

Assume that $m$ is determining. $\mathcal{S}$ basic semi-algebraic defined by $P_j$'s in $\mathscr{D}'$

$(\exists! \mu$ realizing $m$ on $\mathcal{S}) \iff (m$ and all ${}_{P_j}m$ are p.s.d.)

# Our result: sketch of the proof (I)

1. ($m$ determining + p.s.d.) $\Rightarrow$ ($\exists ! \mu$ non-negative on $\mathscr{D}'$ realizing $m$)

$$\langle f^{(n)}, m^{(n)} \rangle = \int_{\mathscr{D}'} \langle f^{(n)}, \gamma^{\otimes n} \rangle \mu(d\gamma).$$

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$$\Downarrow$$

$$\langle f^{(n)}, ({}_{P_j}m)^{(n)}\rangle = \int_{\mathscr{D}'} \langle f^{(n)}, \gamma^{\otimes n}\rangle P_j(\gamma)\mu(d\gamma).$$

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2. ($m$ determining) $\Rightarrow$ (${}_{P_j}m$ determining)

3. (${}_{P_j}m$ determining + p.s.d.)$\Rightarrow$($\exists!\nu$ non-negative on $\mathscr{D}'$ realizing ${}_{P_j}m$)

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