Growth Series of Cyclotomic and Root Lattices

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Polyhedra, Lattices, Algebra, and Moments

IMS Singapore

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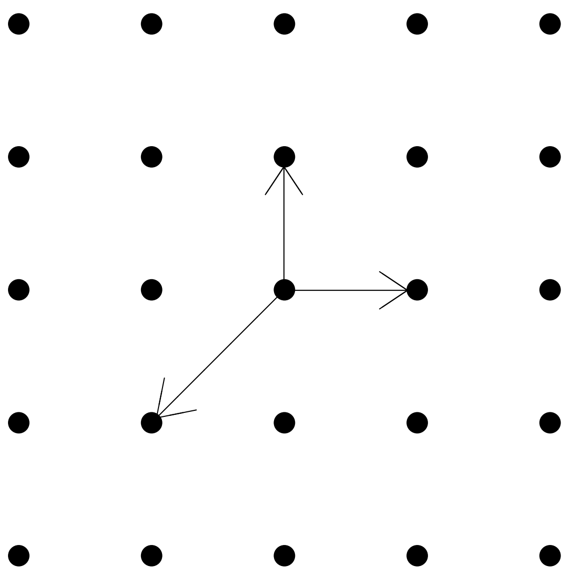
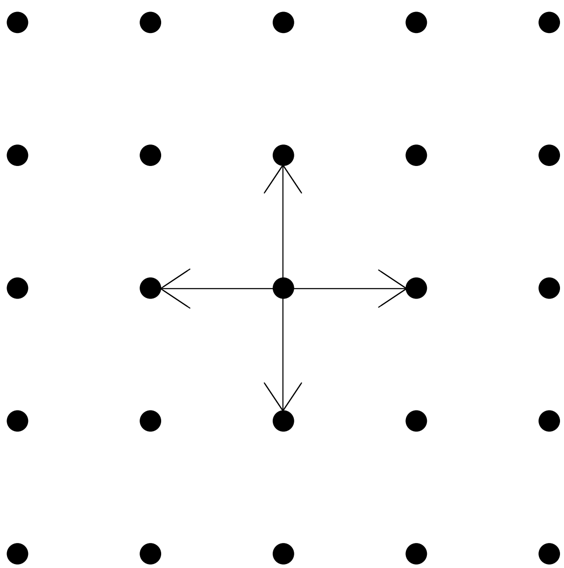
Lattices, Monoid Generators, and Growth Series





Cyclotomic Lattices

$\mathcal{L} = \mathbb{Z}[e^{2\pi i/m}] \cong \mathbb{Z}^{\varphi(m)}$

M – all m^{th} roots of unity (suitably identified in $\mathbb{R}^{\varphi(m)}$)

h_m – coordinator polynomial of $\mathbb{Z}[e^{2\pi i/m}]$

Theorem (Kløve–Parker 1999) The coordinator polynomial of $\mathbb{Z}[e^{2\pi i/p}]$, where p is prime, equals $h_p(x) = x^{p-1} + x^{p-2} + \cdots + 1$.

Conjectures (Parker 1999)

(1) $h_m(x) = g(x)^{\frac{m}{\sqrt{m}}}$ for a palindromic polynomial g of degree $\varphi(\sqrt{m})$.

(2) $h_{2p}(x) = \sum_{k=0}^{\frac{p-3}{2}} \left(x^k + x^{p-1-k}\right) \sum_{j=0}^{k} \binom{p}{j} + 2^{p-1} x^{\frac{p-1}{2}}$

(3) $h_{15}(x) = \left(1 + x^8\right) + 7\left(x + x^7\right) + 28\left(x^2 + x^6\right) + 79\left(x^3 + x^5\right) + 130x^4$

Root Lattices

Theorem (Conway–Sloane, Bacher–de la Harpe–Venkov 1997)

$$h_{A_n}(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

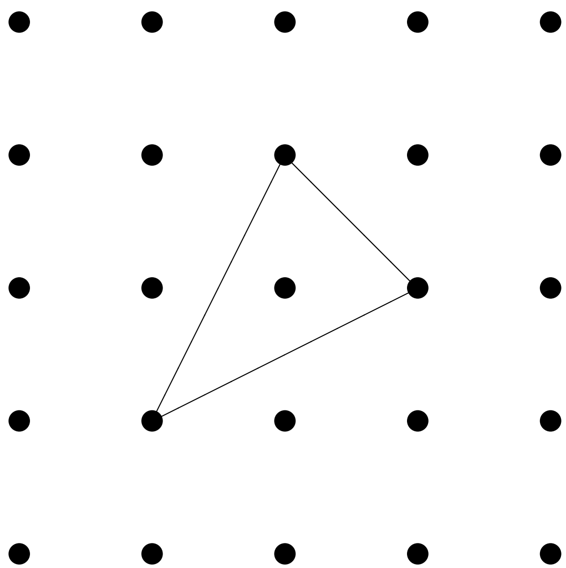
$$h_{B_n}(x) = \sum_{k=0}^{n} \left[\binom{2n+1}{2k} - 2k\binom{n}{k} \right] x^k$$

$$h_{C_n}(x) = \sum_{k=0}^{n} \binom{2n}{2k} x^k$$

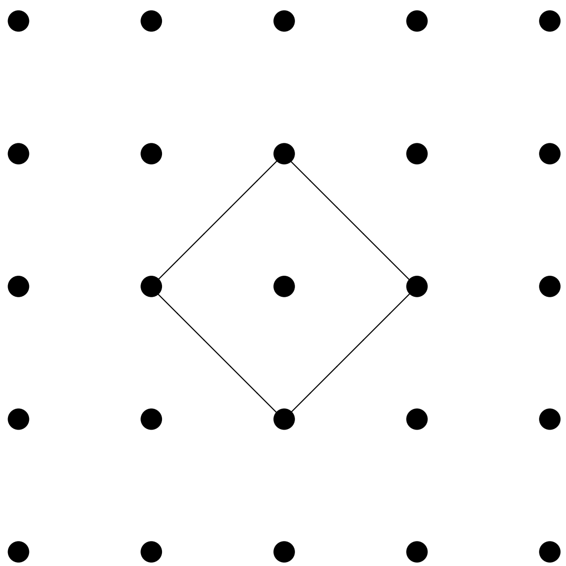
$$h_{D_n}(x) = \sum_{k=0}^{n} \left[\binom{2n}{2k} - \frac{2k(n-k)}{n-1}\binom{n}{k} \right] x^k$$

Capturing Growth Series

$\mathcal{C}_3$



$\mathcal{C}_4 = \mathcal{P}_{D_2}$



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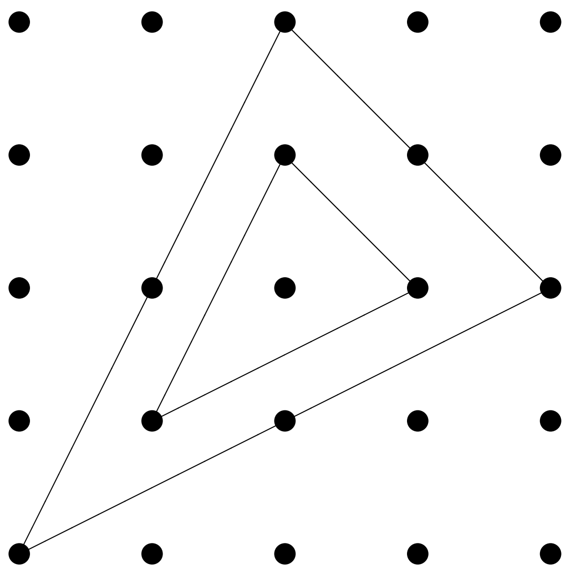

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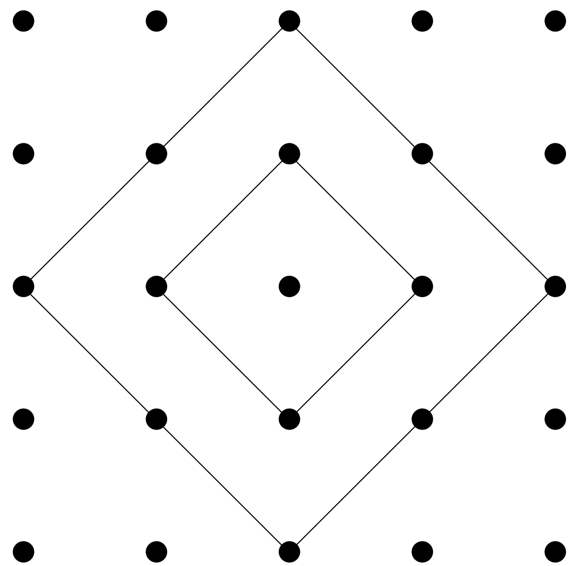

Coordinator Polynomials of Root Lattices

Theorem (Conway–Sloane, Bacher–de la Harpe–Venkov, 1997)

$$
\begin{aligned}
h_{A_n}(x) &= \sum_{k=0}^{n} \binom{n}{k}^2 x^k \\
h_{B_n}(x) &= \sum_{k=0}^{n} \left[\binom{2n+1}{2k} - 2k\binom{n}{k} \right] x^k \\
h_{C_n}(x) &= \sum_{k=0}^{n} \binom{2n}{2k} x^k \\
h_{D_n}(x) &= \sum_{k=0}^{n} \left[\binom{2n}{2k} - \frac{2k(n-k)}{n-1}\binom{n}{k} \right] x^k
\end{aligned}
$$

Theorem (Ardila–MB–Hoşten–Pfeifle–Seashore) The coordinator polynomials of the growth series of root lattices of type A, C, D are the h-polynomials of any unimodular triangulation of the respective polytopes $\mathcal{P}_{A_n}, \mathcal{P}_{C_n}, \mathcal{P}_{D_n}$.

Cyclotomic Polytopes

For two polytopes $P \subset \mathbb{R}^{d_1}$ and $Q \subset \mathbb{R}^{d_2}$, each containing the origin in its interior, we define the direct sum $P \circ Q := \operatorname{conv}(P \times \mathbf{0}_{d_2}, \mathbf{0}_{d_1} \times Q)$. For a prime p, we define the cyclotomic polytope

$$\mathcal{C}_{p^\alpha} = \underbrace{\mathcal{C}_p \circ \mathcal{C}_p \circ \cdots \circ \mathcal{C}_p}_{p^{\alpha-1} \text{ times}} .$$

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For two polytopes $P = \operatorname{conv}(v_1, v_2 \ldots, v_s)$ and $Q = \operatorname{conv}(w_1, w_2, \ldots, w_t)$ we define their tensor product

$$P \otimes Q := \operatorname{conv}(v_i \otimes w_j \ : \ 1 \leq i \leq s, \ 1 \leq j \leq t) .$$

Our construction implies for $m = m_1 m_2$, where $m_1, m_2 > 1$ are relatively prime, that the cyclotomic polytope $\mathcal{C}_m$ is equal to $\mathcal{C}_{m_1} \otimes \mathcal{C}_{m_2}$.

For general m,

$$\mathcal{C}_m = \underbrace{\mathcal{C}_{\sqrt{m}} \circ \mathcal{C}_{\sqrt{m}} \circ \cdots \circ \mathcal{C}_{\sqrt{m}}}_{\frac{m}{\sqrt{m}} \text{ times}}$$

Coordinator Polynomials of Cyclotomic Lattices

Conjectures (Parker 1999)

(1) $h_m(x) = g(x)^{\frac{m}{\sqrt{m}}}$ for a palindromic polynomial g of degree $\varphi(\sqrt{m})$.

(2) $h_{2p}(x) = \sum_{k=0}^{\frac{p-3}{2}} \left(x^k + x^{p-1-k}\right) \sum_{j=0}^{k} \binom{p}{j} + 2^{p-1} x^{\frac{p-1}{2}}$

(3) $h_{15}(x) = \left(1 + x^8\right) + 7\left(x + x^7\right) + 28\left(x^2 + x^6\right) + 79\left(x^3 + x^5\right) + 130x^4$

Theorem (MB–Hoşten) If m is divisible by at most two odd primes then the boundary of the cyclotomic polytope $\mathcal{C}_m$ admits a unimodular triangulation.

Open Problems

- Describe the face structure of $\mathcal{C}_m$, e.g., in the case $m = pq$.
- Is $\mathcal{C}_m$ normal for all m?
- S. Sullivant computed that the dual of $\mathcal{C}_{105}$ is not a lattice polytope, i.e., $\mathcal{C}_{105}$ is not reflexive. If we knew that $\mathcal{C}_{105}$ is normal, a theorem of Hibi would imply that the coordinator polynomial h_{105} is not palindromic, and hence that Parker's Conjecture (1) is not true in general.