Checkerboard discrepancies

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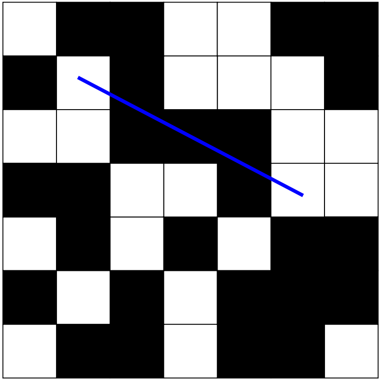
Joint work with Alex Iosevich and Ioannis Parissis

A question



- Imagine the plane as an infinite checkerboard
- Can it be colored black and white so that *any* line segment has almost the same black as white length?
- Can the excess of one color over the other be bounded by a constant?

What we're really interested in

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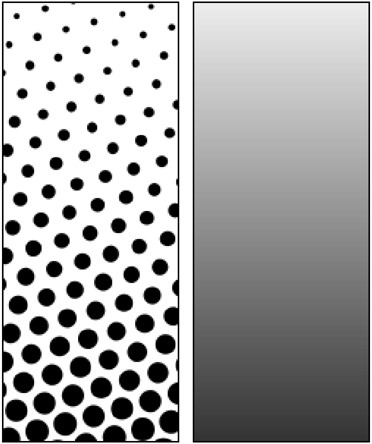
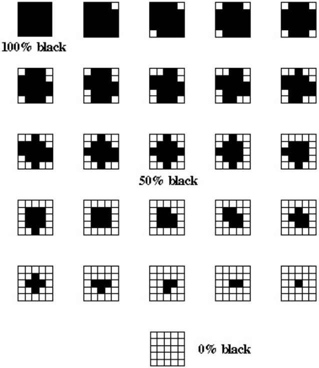
... mostly interested in her hair.

Digital Halftoning



Gray scale value and the halftone screen



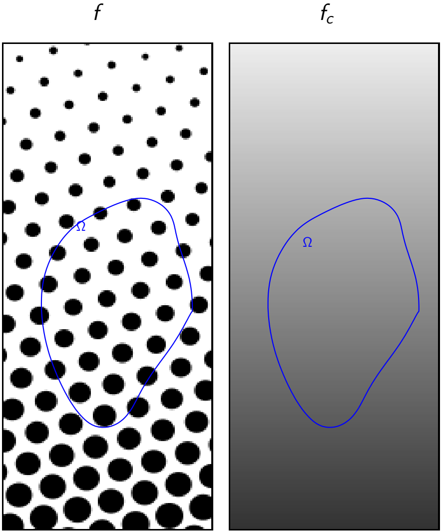
Each square represents a cell in the screen. Each dot represents a spot of ink or laser printer toner.

- Replace continuous grey with a distribution of black and white dots.
- This is how most printers work.

Error committed by halftoning (discrepancy)

- f_c on right takes values in $[0, 1]$.
- f on left is binary.
- For region Ω the *discrepancy* is

$$D(\Omega) = \left| \int_\Omega f - \int_\Omega f_c \right|$$

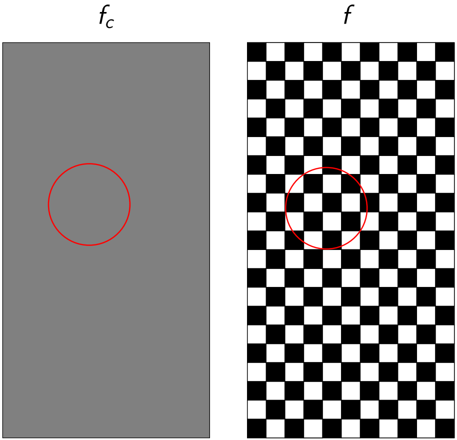


Discrepancy of a family of regions

- Take $f_c = \frac{1}{2}$ everywhere
- Fix binary approximation f
- For Ω in a family $\mathcal{F}$ of regions how large can the discrepancy be?

$$D(\mathcal{F}) = \sup_{\Omega \in \mathcal{F}} D(\Omega)$$

- For instance:
  Ω can be all translates of a disk of fixed radius



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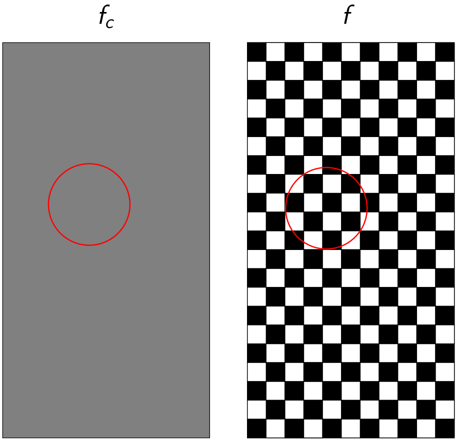

- **Lower bounds:**
  No matter what f is the discrepancy of a family will be large.
- **Upper bounds:**
  There is (also: find) f with discrepancy as small as possible.

Classical geometric discrepancy

- How well can a point distribution approach uniformity?

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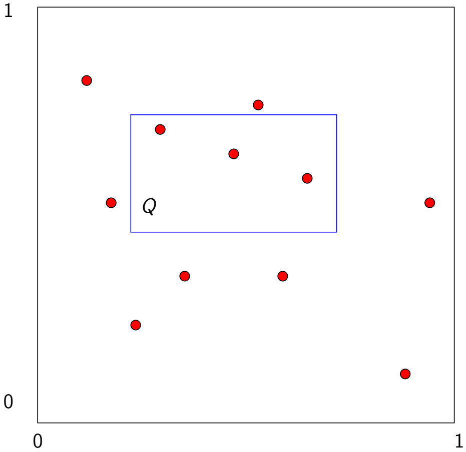
- If Q is any aligned rectangle with $n = |\mathcal{P} \cap Q|$ how large must $|n - |Q| \cdot N|$ be?
- $\sim \log N$ is the answer here (W. Schmidt).



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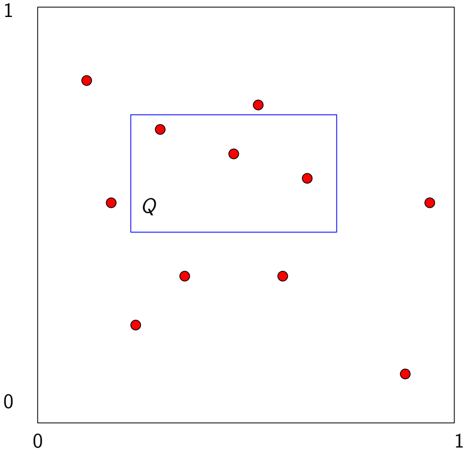

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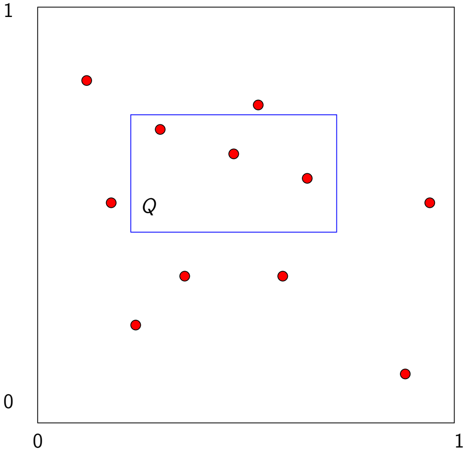

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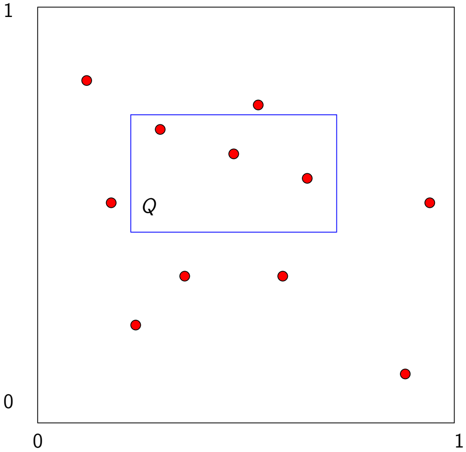
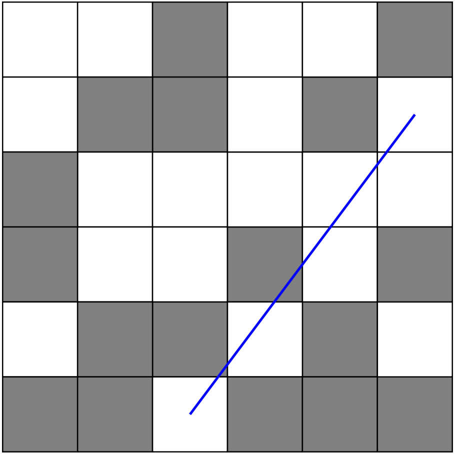


Other families:

- **Anchored rectangles**: Same as translating aligned rectangles
- **Rotating rectangles**: $\sim N^{1/4}$ up to logarithms
- **Disks**: $\sim N^{1/4}$ up to logarithms

A needle on a checkerboard

- The $N \times N$ checkerboard is colored black & white.
- How large is the discrepancy of line segments?
  How much does the white part differ from the black?
- Any length, any placement.



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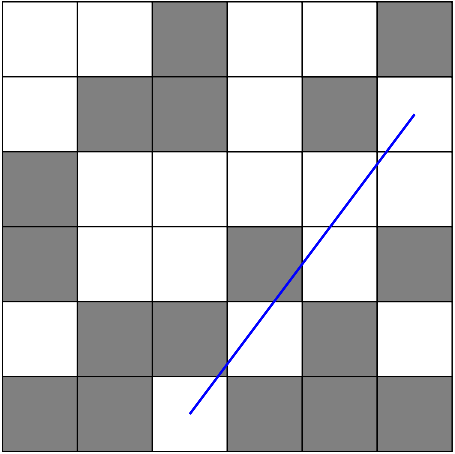


Upper bound:

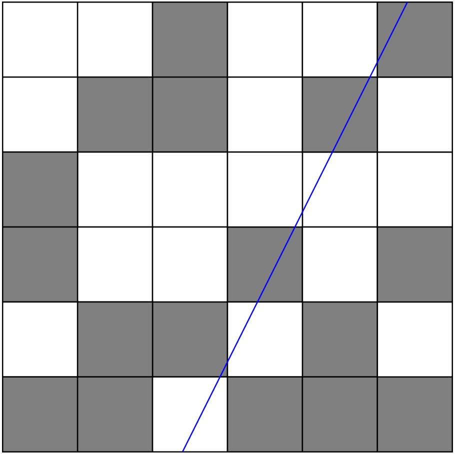
- **Random coloring**: Discrepancy is $O(\sqrt{N \log N})$.
- **Quasi-Random coloring**: Discrepancy of length L is $O(L^{\frac{1}{2}+\epsilon})$, any $\epsilon > 0$, for $L \geq 1$.

Discrepancy lower bound for needle of length L: $\gtrsim \sqrt{L}$

- Enough for lines spanning the $N \times N$ board.

$$\gtrsim \sqrt{N}$$

- Fourier analytic proof



A Fourier lemma for the checkerboard function

- The *checkerboard function*: $f : \mathbb{R}^2 \to \{0, \pm 1\}$:

$$f \equiv 0 \text{ off } [0, N]^2,$$

and

$$f \equiv \pm 1 \text{ in cell } [i, i+1) \times [j, j+1), \quad i, j = 0, 1, \ldots, N-1.$$

- **Lemma:** If A is sufficiently large and a sufficiently small constants

$$\int_{\frac{a}{N} \le |\xi| \le A} \left| \widehat{f}(\xi) \right|^2 d\xi \ge \frac{1}{3} N^2 = \frac{1}{3} \|f\|_2^2.$$

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- **Intuitevely:** *Can discard*

wavelengths $\gtrsim N$ *and* $\lesssim 1$

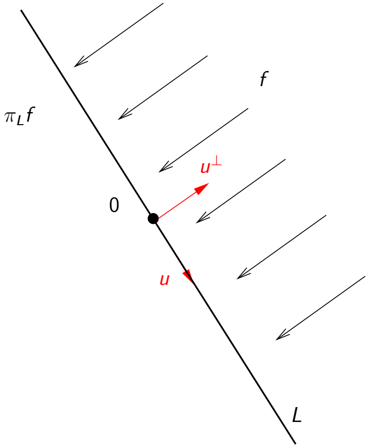
and still keep a constant fraction of the energy

Proof of the lower bound $\sqrt{N}$

- Project f onto line L through origin:

$$\pi_L f(t) = \int_{\mathbb{R}} f(tu + su^{\perp})\, du$$

- Then $\widehat{\pi_L f}(\xi) = \widehat{f}(\xi u)$



Proof of the lower bound $\sqrt{N}$, continued

Define $M = \sup_{L,t} |\pi_L f(t)|$. Must show $M \gtrsim \sqrt{N}$.

$$N^2 = \int f^2 = \int \left|\widehat{f}\right|^2$$

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&\lesssim A M^2 N \quad \text{(since } \operatorname{diam} \operatorname{supp} f \sim N\text{)}
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Hence $M^2 \gtrsim N$

Circular arcs with large discrepancy

- For any curve C discrepancy is

$$\left|\int_C f\right|$$

- We show there is a circle C of radius

$$\frac{1}{5}N \le R \le \frac{1}{4}N$$

such that

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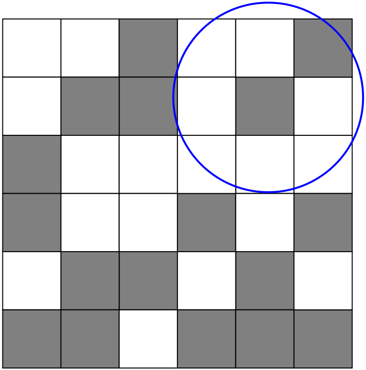

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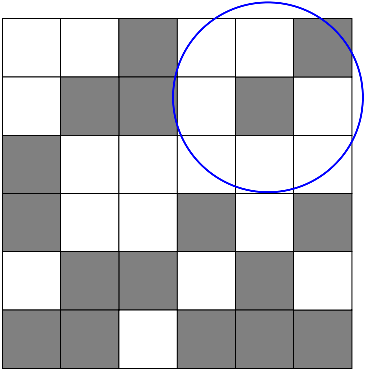

- This does not give a whole circle contained in the $N \times N$ square with large discrepancy, only an arc.
- Our circles are free to translate and dilate

Fourier transform of circle measure

- σ_t is arc-length measure on circle of center 0, radius t.
- We have $\widehat{\sigma_t}(\xi) = t\widehat{\sigma_1}(t \cdot \xi)$

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  where $r = |\xi| \to \infty$.
- **Fourier Lemma:** If $c_0 > 0, c_1 > 1$ then there is $c_2 > 0$ such that

$$\int_x^{c_1 x} \left|\widehat{\sigma_1}(\xi)\right|^2 d\xi \geq c_2. \quad (\text{for } x > c_0)$$

Circle discrepancy lower bound: the L^2 approach, again

- $C(x,t)$ is the circle of center x, radius t.
  Discrepancy function: $D_t(x) = \int_{C(x,t)} f = f * \sigma_t(x)$.

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- We bound from below the L^2 norm

$$\int_{\alpha N}^{\beta N} \int_{\mathbb{R}^2} |D_t(x)|^2 \, dx \, dt \gtrsim N^4$$

  where $\alpha = \frac{1}{5}, \beta = \frac{1}{4}$.

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- Since $\operatorname{diam} \operatorname{supp} D_t(\cdot) \lesssim N$ it follows that

$$\sup_{x,t} |D_t(x)|^2 \gtrsim N$$

L^2 lower bound for the circle discrepancy

$$\int_{\alpha N}^{\beta N} \int_{\mathbb{R}^2} \left|D_t(x)\right|^2 dx\, dt \quad = \quad \int_{\alpha N}^{\beta N} \int_{\mathbb{R}^2} \left|\widehat{D_t}(\xi)\right|^2 d\xi\, dt$$

L^2 lower bound for the circle discrepancy

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&\geq \int_{\alpha N}^{\beta N} \int_{\frac{a}{N} \leq |\xi| \leq A} \left| \widehat{D_t}(\xi) \right|^2 d\xi \, dt \\
&= \int_{\frac{a}{N} \leq |\xi| \leq A} \left| \widehat{f}(\xi) \right|^2 \int_{\alpha N}^{\beta N} |\widehat{\sigma}_t(\xi)|^2 \, dt \, d\xi
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&\ge \alpha^2 N^2\int_{\frac{a}{N}\le|\xi|\le A}\left|\widehat{f}(\xi)\right|^2\frac{1}{|\xi|}\int_{\alpha|\xi|N}^{\beta|\xi|N}\left|\widehat{\sigma_1}(u)\right|^2\,du\,d\xi
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&\ge \alpha^2 N^2 \int_{\frac{a}{N} \le |\xi| \le A} \left|\widehat{f}(\xi)\right|^2 \frac{1}{|\xi|} \int_{\alpha |\xi| N}^{\beta |\xi| N} |\widehat{\sigma_1}(u)|^2 \, du \, d\xi \\
&\ge C \frac{\alpha^2 N^2}{A} \int_{\frac{a}{N} \le |\xi| \le A} \left|\widehat{f}(\xi)\right|^2
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&\ge \alpha^2 N^2 \int_{\frac{a}{N}\le|\xi|\le A} \left|\widehat{f}(\xi)\right|^2 \frac{1}{|\xi|} \int_{\alpha|\xi|N}^{\beta|\xi|N} |\widehat{\sigma_1}(u)|^2 \, du\, d\xi \\
&\ge C\frac{\alpha^2 N^2}{A} \int_{\frac{a}{N}\le|\xi|\le A} \left|\widehat{f}(\xi)\right|^2 \\
&\ge CN^4.
\end{aligned}
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Different norms

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- They translate to lower bounds for the sup norm.

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- Other norms are possible and often studied in classical discrepancy
- The L^p norm

  Line discrepancy: two parameters, angle u and x-intercept

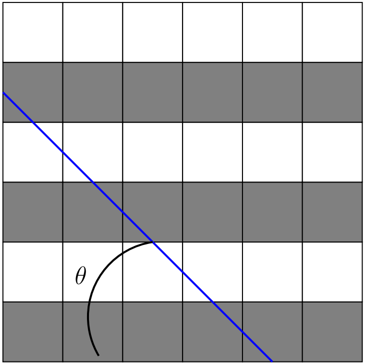
$$\Delta(f,p) = \left(\frac{1}{N} \int_{S^1} \int |\Delta(u,x)|^p \, dx \, du \right)^{1/p}$$

  Circle discrepancy: 3 parameters, center x and radius t

$$D(f,p) = \left(\frac{1}{N^3} \int_{N/5}^{N/4} \int |D_t(x)|^p \, dx \, dt \right)^{1/p}$$

- Essentially increasing in p

Different norms: things are different for L^1



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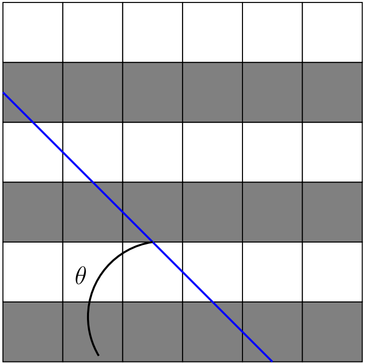

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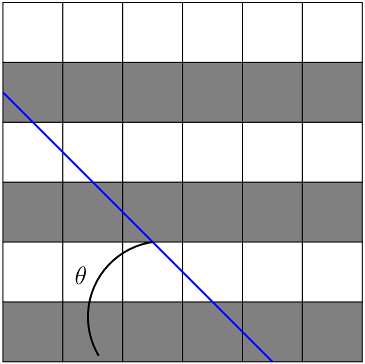

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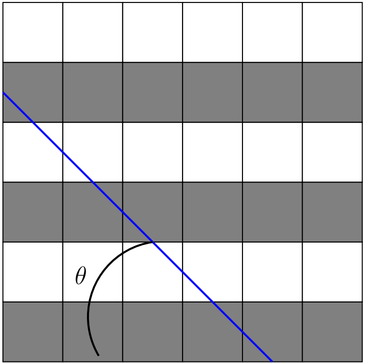

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- No lower bound is known but $\log N$ is probably the correct order.

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- Working as in the case of variable t we get

$$\int_{|x| \lesssim N} |D_t(x)|^2 + |D_{2t}(x)|^2 \, dx \gtrsim N^2 t$$

- So for any $t \lesssim N$ there exists x such that

$$|D_t(x)| \gtrsim \sqrt{t} \text{ or } |D_{2t}(x)| \gtrsim \sqrt{t}$$

Circle discrepancy: fixing the radius completely

- Fix radius $t \sim N$ and a coloring f. Then there is a circle C of radius t with
$$\left|\int_C f\right| \gtrsim \sqrt{t}$$
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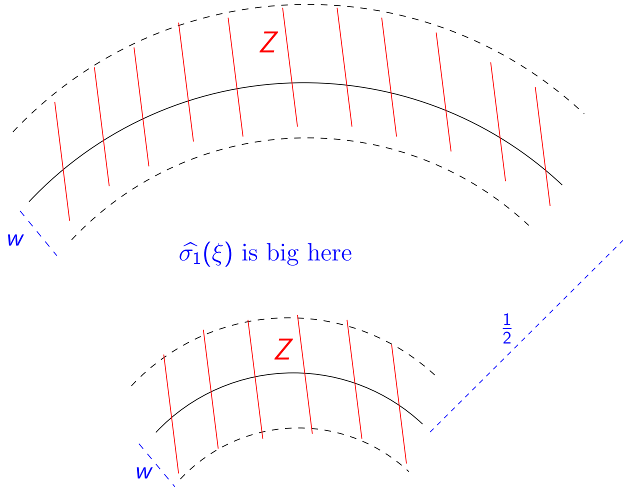
- Plan is
  - throw away neighborhood of the roots of $\widehat{\sigma_t}$
  - show $\int \left|\widehat{f}\right|^2$ has not lost much.

Ignoring where $\widehat{\sigma_1}(\xi)$ is small



Asymptotics for $\widehat{\sigma_1}(\xi)$ tell us

- Root circles are spaced roughly by $\frac{1}{2}$
- Staying constant distance w from them guarantees $|\widehat{\sigma_1}(\xi)|^2 \gtrsim \frac{1}{|\xi|}$

A corresponding Poincaré type inequality

- Let Z be the region crossed out (annuli).
- Then for $g \in C^1(\mathbb{R}^2)$ we have

$$\int |g|^2 \lesssim \int_{Z^c} |g|^2 + w^2 \int |\nabla g|^2$$

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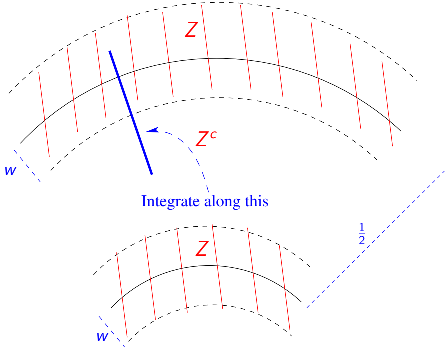
- Consequence of the bounded geometry
- Integrate along the radius. Essentially the one-dimensional

$$\int_{1/2}^{1} |g|^2 \lesssim \int_0^{1/2} |g|^2 + \int_0^1 |g'|^2$$



Proof of the lower bound for fixed radius

$$\int_{\mathbb{R}^2} |D_t(x)|^2 \, dx = \int_{\mathbb{R}^2} \left|\widehat{f}(\xi)\right|^2 |\widehat{\sigma}_t(\xi)|^2 \, d\xi = \int_{\mathbb{R}^2} \left|\widehat{f}(\frac{u}{t})\right|^2 |\widehat{\sigma_1}(u)|^2 \, du$$

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\end{aligned}$$

But $\int \left|\nabla\widehat{f}(\xi)\right|^2 d\xi = \||x|f(x)\|_2^2 \lesssim N^4$ and $\int \left|\widehat{f}(\xi)\right|^2 d\xi = \|f\|_2^2 = N^2$.
Since $t \sim N$, choose w a small constant to get

$$\frac{1}{N^2}\int |D_t(x)|^2\,dx \gtrsim t$$

Thank you for your attention.