Weighted Ehrhart polynomials and intermediate sums on polyhedra

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with
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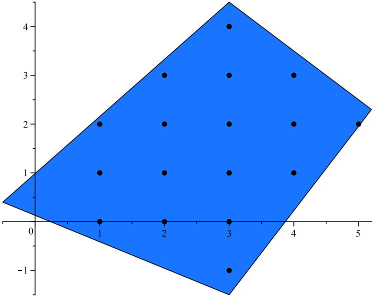
Weighted sums on lattice points of a polytope $\mathfrak{p} \subset \mathbb{R}^d$



$\mathfrak{p} \subset \mathbb{R}^d$ convex semi-rational polytope. ($Ax \leq b$, A rational matrix, $b \in \mathbb{R}^N$).
Weight $h(x) :=$ polynomial function on $\mathbb{R}^d$.
Sum of values $h(x)$ over set of lattice points in $\mathfrak{p}$.

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

Number of lattice points

If the dimension is part of input, computing number of points is $\sharp P$-hard (even for a simplex).

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Volume is an approximation, when $\mathfrak{p}$ is dilated.
$\mathrm{Card}(t\mathfrak{p} \cap \mathbb{Z}^d) \sim \mathrm{vol}(\mathfrak{p}) t^d$ when $t \to +\infty$.
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Ehrhart quasi-polynomial (1962) $\mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d) =$

$$\mathrm{vol}(\mathfrak{p}) n^d + E_{d-1}(n) n^{d-1} + \cdots + E_{d-k}(n) n^{d-k} + \cdots + E_0(n)$$

Coefficients are periodic functions of n (constant if $\mathfrak{p}$ has integral vertices).

Example $\mathfrak{p} = [0, \frac{1}{2}]$. Number of lattice points is $\frac{n}{2} + 1$ for n even, $\frac{n}{2} + \frac{1}{2}$ for n odd.

Computation of the number of points

Theorem (Barvinok 1994)

If $\mathfrak{p}$ is an rational polytope of fixed dimension d, then its number of lattice points can be computed in polynomial time.

First implementation: LattE, (De Loera, Hemmecke, Tauzer, and Yoshida, 2003)

Theorem (Barvinok 2005)

If $\mathfrak{p}$ is a rational simplex,($\dim \mathfrak{p}$ is part of the input), if k is fixed, then the k highest degree coefficients of the Ehrhart quasi-polynomial can be computed in polynomial time.

$$\mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d) = \mathrm{vol}(\mathfrak{p}) n^d + E_{d-1}(n) n^{d-1} + \cdots + E_{d-k}(n) n^{d-k} + \cdots$$

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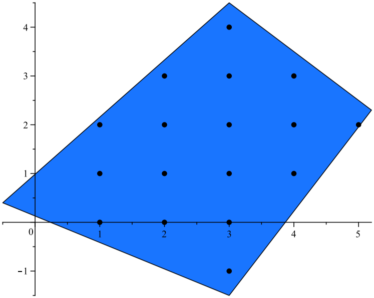
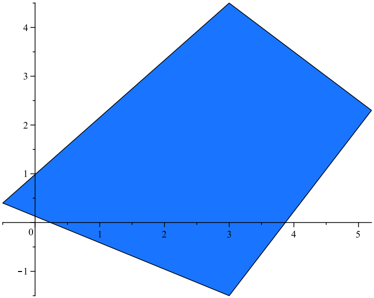
Main idea:
highest degree terms can be described and computed by means of remarkable
linear combinations of intermediate sums.

Lattice points

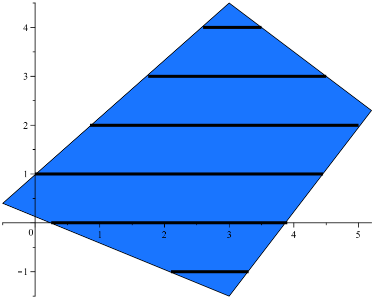


Integral over polytope



Intermediate sums

$L \subseteq \mathbb{R}^d$ fixed rational subspace.
Slice $\mathfrak{p}$ by parallels to L through lattice points and add the integrals over the slices.



Intermediate sums

$$S^L(\mathfrak{p}) := \sum_y \mathrm{vol}(\mathfrak{p} \cap (y + L))$$

Weight $h(x)$,

$$S^L(\mathfrak{p}, h) := \sum_y \int_{\mathfrak{p}\cap(y+L)} h(x)dx$$

summation index y runs over projected lattice in $\mathbb{R}^d/L$.

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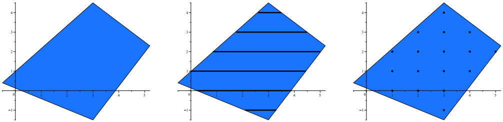
$L = \mathbb{R}^d \to$ integral of the weight over $\mathfrak{p}$.
$L = \{0\} \to$ weighted sum over lattice points of $\mathfrak{p}$.

Polytime computation of intermediate sums on a simplex

$S^L(\mathfrak{p}, h) :=$



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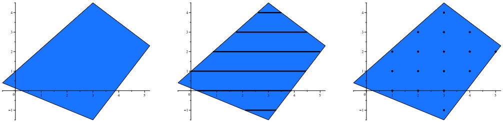

Theorem Computing integral of h on a simplex NP-hard if deg h not fixed. ("How to integrate a polynomial on a simplex" B.B.DeL.K.V. 2008).

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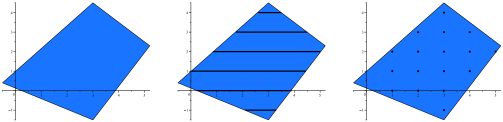

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Theorem. $\mathfrak{p}$ a simplex. *Assume* deg h *fixed, or* $h(x) = \ell(x)^m$ *(more generally* $P(\ell_1(x), \ldots, \ell_N(x))$, *N fixed,* $\ell_j(x)$ *linear).*

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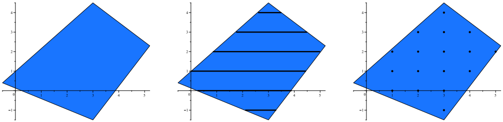

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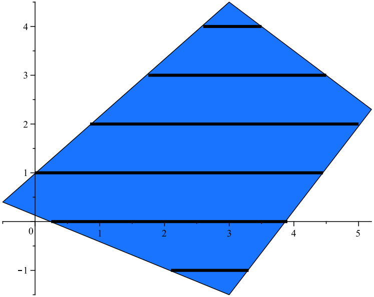
• *Then,* $\int_{\mathfrak{p}} h(x)dx$ *can be computed in polynomial time. (B.B.DeL.K.V. 2008)*

• *If , moreover,* codim $L \leq k$ *with* k *fixed,* $S^L(\mathfrak{p}, h)$ *can be computed in polynomial time.* (unweighted: Barvinok, 2005, weighted: B.B.DeL.K.V. 2010)



Proof: discrete sums in dimension $\leq k$, integrals in dimension $d-k$.

Ehrhart quasi-polynomials for intermediate sums

Weight $h(x)$,

$$S(t\mathfrak{p}, h) := \sum_{x \in t\mathfrak{p} \cap \mathbb{Z}^d} h(x), \quad S^L(t\mathfrak{p}, h) := \sum_{y} \int_{t\mathfrak{p} \cap (y+L)} h(x)dx$$

summation index y runs over projected lattice in $\mathbb{R}^d/L$.

For a *dilated* *polytope* $t\mathfrak{p}$*, intermediate sum is a* *quasi-polynomial* *function of the parameter* t*,*

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summation index y runs over projected lattice in $\mathbb{R}^d/L$.

For a dilated polytope $t\mathfrak{p}$, intermediate sum is a quasi-polynomial function of the parameter t, the highest degree coefficient is the integral of the weight.
unweighted, $t \in \mathbb{N}$: Barvinok 2005,
weighted, t real: BBDeLKV 2010

Barvinok's linear combination

$\mathcal{L}$ finite family of subspaces L.

$\rho(L) \in \mathbb{Z}$.

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})$$

Fix $k \leq \dim \mathfrak{p}$.

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`drawings on blackboard`

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- family $\mathcal{L}$ contains the linear spaces parallel to all faces of codimension $\leq k$ of $\mathfrak{p}$,
- family $\mathcal{L}$ is closed under sum,
  `drawings on blackboard`
- coefficients satisfy Moebius property

$$[\bigcup_{L \in \mathcal{L}} L^\perp] = \sum_{L \in \mathcal{L}} \rho(L)[L^\perp].$$

$L^\perp$ orthogonal of L in dual space of $\mathbb{R}^d$

Top degree coefficients of Ehrhart quasi-polynomial

Theorem. Barvinok 2005

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$\mathcal{L}$ family of subspaces, containing the linear spaces parallel to all faces of codimension $\leq k$ of $\mathfrak{p}$ and closed under sum, coefficients $\rho(L)$ satisfying Moebius property.
Then the quasi-polynomials

$$S(n\mathfrak{p}) \text{ and } \sum_{L\in\mathcal{L}} \rho(L) S^L(n\mathfrak{p})$$

have the same coefficients of degree $\geq d - k$

$$\operatorname{vol}(\mathfrak{p}) n^d + \cdots + E_{d-k}(n) n^{d-k}.$$

Furthermore, if $\mathfrak{p}$ is a simplex, $\dim \mathfrak{p} \in$ input, then $\sum_{L\in\mathcal{L}} \rho(L) S^L(n\mathfrak{p})$ can be computed in polynomial time

Faces poset and its Moebius coefficients

Dimension 2.
Triangle with vertices $(1,2,3)$. Faces of codimension ≤ 1
3 edges of codimension 1 $\rightarrow \rho(L_{(i,j)}) = 1$
Whole space $L_{(1,2,3)} = \mathbb{R}^2 \rightarrow \rho(L_{(1,2,3)}) = -2$

`Drawings`

Dimension 3.
Tetrahedron with vertices $(1,2,3,4)$.
Faces of codimension ≤ 2 and their sums.
4 edges of codimension 2 $\rightarrow \rho(L_{(i,j)}) = 1$
4 facets of codimension 1 $\rightarrow \rho(L_{(i,j,k)}) = -2$
3 pairs of opposite edges $\rightarrow \rho(L_{(i,j),(k,l)}) = -1$
Whole space $L_{(1,2,3,4)} = \mathbb{R}^3 \rightarrow \rho(L_{(1,2,3,4)}) = 6$

Main ideas for number of points in fixed dimension

- Generating functions for polyhedra.

$$S(\mathfrak{p})(\xi) := \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} \mathrm{e}^{\langle \xi, x \rangle}, \text{ meromorphic function of } \xi.$$

  Example $[s, +\infty[$, $a := \lceil s \rceil$, $\sum_{n \geq a} \mathrm{e}^{n\xi} = \frac{\mathrm{e}^{a\xi}}{1-\mathrm{e}^{\xi}}$.
- If $\mathfrak{p}$ is compact, number of lattice points $=$ $S(\mathfrak{p})(0)$.

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- If $\mathfrak{p}$ is compact, number of lattice points $=$ $S(\mathfrak{p})(0)$.
- Brion's theorem. $\mathfrak{p}$ polytope. For each vertex s, $C_s :=$ cone of feasible directions at vertex s,

$$S(\mathfrak{p})(\xi) = \sum_s S(s + C_s)(\xi).$$ Poles cancel out in sum over vertices.

Example: `Drawing`. For a and b integers,
$\sum_{n=a}^{\infty} e^{n\xi} + \sum_{n=-\infty}^{b} e^{n\xi} =$

$$\frac{e^{a\xi}}{1 - e^{\xi}} + \frac{e^{b\xi}}{1 - e^{-\xi}} = \frac{e^{a\xi} - e^{(b+1)\xi}}{1 - e^{\xi}} = \sum_{n=a}^{b} e^{n\xi}$$

- Short formulas for generating functions of unimodular cones (generated by lattice vectors $v_1, \ldots, v_d$ with $\det(v_j) = 1$).

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- Barvinok's decomposition. Any convex polyhedral cone can be decomposed into a sum of unimodular cones, in polynomial time when dimension is fixed. (Short vector in a lattice, Minkowski, LLL.)

Our results. (B.,B.,De L.,K.,V., 2008-2013)

- Ehrhart theory for weighted case and real parameter t
  $S^L(t\mathfrak{p}, h) = (\int_{\mathfrak{p}} h)\, t^{d+\deg h} + \cdots + E_{d+\deg h-k}(t) t^{d+\deg h-k} + \cdots$
  Requires to describe coefficients as *step-polynomials* of t, instead of pointwise computation of $E_k(n)$ for $n \in \mathbb{N}$

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  Requires to describe coefficients as *step-polynomials* of t, instead of pointwise computation of $E_k(n)$ for $n \in \mathbb{N}$
- Given polytope $\mathfrak{p}$, weight $h(x)$ and codimension k, we canonically define two quasi-polynomials
  $\mathrm{Barvinok}_{\mathfrak{p},h,k}(t)$ (similar to Barvinok's approximation of the number of points)
  $\mathrm{ConeByCone}_{\mathfrak{p},h,k}(t)$ (new and simpler)

- We prove that $S(t\mathfrak{p}, h)$, $\text{Barvinok}_{\mathfrak{p},h,k}(t)$, and $\text{ConeByCone}_{\mathfrak{p},h,k}(t)$ have the same terms of degree $\geq d + \deg(h) - k$.

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- Similar results for semi-rational parametric polytopes $Ax \leq b$, A fixed rational matrix, parameter $b \in \mathbb{R}^N$.
- We describe polynomial-time algorithms for computing $\text{Barvinok}_{\mathfrak{p},h,k}(t)$ and $\text{ConeByCone}_{\mathfrak{p},h,k}(t)$, if
  k is fixed,
  $\mathfrak{p}$ a semi-rational simplex, (also holds for more general data),
  weight h has fixed degree, or $h(x) = \ell(x)^m$ (more generally $P(\ell_1(x), \dots, \ell_N(x))$, N fixed, $\ell_j(x)$ linear).
  (implemented in a Maple program, see *LattE integrale*).

Methods might have their own interest

- Intermediate generating functions of a polyhedron $\mathfrak{q}$. $S^L(\mathfrak{q})(\xi) := \sum_y \int_{\mathfrak{q}\cap(y+L)} \mathrm{e}^{\langle \xi, x\rangle}\, \mathrm{d}x$.
  We prove an approximation theorem for generating functions (in the sense of power series approximation)of affine convex polyhedral cones,
  from which we derive equality of highest degree terms of quasi-polynomials.
- All this relies on properties of intermediate generating function $S^L(s+C)(\xi)$ of a shifted cone $s+C$ with vertex s, as function of the pair (s,ξ).
- Icing on the cake. A beautiful formula for the Moebius coefficients of a simplex. (discovered in *A. Bjorner and L. Lovasz, Linear decision trees, subspace arrangements and Moebius functions*, 1994, where it is attributed to Stanley)

Case $h(x) = \frac{\langle \ell, x\rangle^m}{m!}$

Weight is a power of a linear form

Weights which are powers of linear forms are related to homogeneous components of Taylor series of generating functions.

$$\int_{\mathfrak{p}} \frac{\langle \ell, x\rangle^m}{m!} dx = \left(\int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle} \right)_{[m]} \Big|_{\xi=\ell}$$

$$\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} \frac{\langle \ell, x\rangle^m}{m!} = \left(\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} \mathrm{e}^{\langle \xi, x\rangle} \right)_{[m]} \Big|_{\xi=\ell}$$

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General case: write $h(x)$ as a linear combination of powers of linear forms.

Intermediate generating functions of polyhedra

$L \subseteq \mathbb{R}^d$ rational subspace, $\mathfrak{q} \subseteq \mathbb{R}^d$ polyhedron (semi-rational convex).

$$S^L(\mathfrak{q})(\xi) := \sum_{y \in \Lambda_{\mathbb{R}^d/L}} \int_{\mathfrak{q} \cap (y+L)} \mathrm{e}^{\langle \xi, x \rangle} \, \mathrm{d}x.$$

for those ξ where integral and series converge.
$S^L(\mathfrak{q})(\xi)$ admits a meromorphic continuation to $\mathbb{C}^d$.

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$S^L(\mathfrak{q})(\xi)$ admits a meromorphic continuation to $\mathbb{C}^d$.
Example: For a and b integers,
$\sum_{n=a}^{\infty} e^{n\xi} = \frac{e^{a\xi}}{1-e^{\xi}}$
$\sum_{n=-\infty}^{b} e^{n\xi} = \frac{e^{b\xi}}{1-e^{-\xi}}$
Meromorphic continuation:
first sum converges for $\xi < 0$, second sum converges for $\xi > 0$.

Brion's theorem

Theorem $\mathfrak{p}$ *polytope. For each vertex s of $\mathfrak{p}$, C_s cone of feasible directions at s, $s + C_s$ supporting cone at s,*

$$S^L(\mathfrak{p})(\xi) = \sum_s S^L(s + C_s)(\xi)$$

Originally, for $L = \{0\}$, proved by Brion using localization in equivariant cohomology of toric varieties.

Combinatorial proof can be deduced from set-theoretic Brianchon-Gram theorem (strengthening of Euler identity).

Hyperplane singularities
Homogeneous components with respect to ξ

Dimension 1. $C = [0, +\infty[$, $s \in \mathbb{R}$,
shifted affine cone $s + C = [s, +\infty[$, $a := \lceil s \rceil$

$$I(s+C)(\xi) := \int_s^{+\infty} e^{\xi x} dx = -\frac{e^{s\xi}}{\xi}.$$

$$S(s+C)(\xi) = \sum_{n \geq a} e^{n\xi} = \frac{e^{a\xi}}{1-e^{\xi}} = \left(\frac{e^{a\xi}\xi}{1-e^{\xi}}\right)\frac{1}{\xi}$$

$$= -\frac{1}{\xi} - \sum_{m \geq 0} \frac{B_{m+1}(a)}{(m+1)!}\xi^m$$

Bernoulli polynomials $\frac{e^{u\xi}\xi}{e^{\xi}-1} = 1 + \sum_{n\geq 1} \frac{B_n(u)}{n!}\xi^n$

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$$\begin{aligned} I(s+C)(\xi) &:= \int_s^{+\infty} \mathrm{e}^{\xi x} dx = -\frac{\mathrm{e}^{s\xi}}{\xi}. \\ S(s+C)(\xi) &= \sum_{n \geq a} \mathrm{e}^{n\xi} = \frac{\mathrm{e}^{a\xi}}{1-\mathrm{e}^{\xi}} = \left(\frac{\mathrm{e}^{a\xi}\xi}{1-\mathrm{e}^{\xi}}\right)\frac{1}{\xi} \\ &= -\frac{1}{\xi} - \sum_{m \geq 0} \frac{B_{m+1}(a)}{(m+1)!}\xi^m \end{aligned}$$

Bernoulli polynomials $\frac{\mathrm{e}^{u\xi}\xi}{\mathrm{e}^{\xi}-1} = 1 + \sum_{n \geq 1} \frac{B_n(u)}{n!}\xi^n$

$S(s+C)(\xi)$ is the (infinite) sum of homogeneous components of degree ≥ -1 , the lowest degree one is $I(C)(\xi) = -\frac{1}{\xi}$.

Hyperplane singularities
Homogeneous components with respect to ξ

We prove similar properties for all intermediate generating functions of cones.

- Can assume C simplicial, edge generators v_i, $1 \leq i \leq d$,

$$S^L(s+C)(\xi) = \frac{\textit{holomorphic at } \xi = 0}{\prod_{i=1}^{d} \langle \xi, v_i \rangle}$$

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$$S^L(s+C)(\xi) = \frac{\textit{holomorphic at } \xi = 0}{\prod_{i=1}^d \langle \xi, v_i \rangle}$$

- $\Rightarrow S^L(s+C)(\xi) =$ sum of series of homogeneous components

$$S^L(s+C)(\xi) = \frac{\textit{Taylor series of numerator}}{\prod_{i=1}^d \langle \xi, v_i \rangle} = \sum_{m \geq -d} S^L(s+C)_{[m]}(\xi)$$

$S^L(s+C)_{[m]}(\xi)$ is a rational function of total degree m.

Examples. $d = 2$, $C = \mathbb{R}^2_{\geq 0}$.

$\{u\} :=$ fractional part of $u \in \mathbb{R}$. $s = (s_1, s_2)$, $\xi = (\xi_1, \xi_2)$.

$S^L(s + C)(\xi) =$

$$L = \mathbb{R}^2, \qquad \mathrm{e}^{\langle \xi, s\rangle} \frac{1}{\xi_1 \xi_2}$$

$$L = \{0\}, \qquad \mathrm{e}^{\langle \xi, s\rangle} \frac{\mathrm{e}^{\{-s_1\}\xi_1} \mathrm{e}^{\{-s_2\}\xi_2}}{(1 - \mathrm{e}^{\xi_1})(1 - \mathrm{e}^{\xi_2})}$$

$$L = \mathbb{R}(1, -1), \qquad \mathrm{e}^{\langle \xi, s\rangle} \left(\frac{\mathrm{e}^{\{-(s_1+s_2)\}\xi_1}}{1 - \mathrm{e}^{\xi_1}} - \frac{\mathrm{e}^{\{-(s_1+s_2)\}\xi_2}}{1 - \mathrm{e}^{\xi_2}}\right)\left(\frac{1}{\xi_1 - \xi_2}\right)$$

$$L = \mathbb{R}(1, 1), \qquad \mathrm{e}^{\langle \xi, s\rangle} \left(\frac{\mathrm{e}^{\{s_2 - s_1\}\xi_1}}{1 - \mathrm{e}^{\xi_1}} - \frac{\mathrm{e}^{-\{s_2 - s_1\}\xi_2}}{1 - \mathrm{e}^{-\xi_2}}\right)\left(\frac{-1}{\xi_1 + \xi_2}\right)$$

Easy to check: all four of the form $\frac{\textit{holomorphic at } \xi = 0}{\xi_1 \xi_2}$,

$$\frac{1}{\xi_1 \xi_2} + \sum_{m \geq -1} \left(\sum_{m_1 + m_2 = m} a_{m_1, m_2} \xi_1^{m_1} \xi_2^{m_2} \right)$$

Weight $h(x) = \frac{\langle \ell, x \rangle^m}{m!}$

$$S^L(\mathfrak{p})\left(\frac{\ell^m}{m!}\right) = \left(\sum_{s \text{ vertex of } \mathfrak{p}} S^L(s + C_s)_{[m]}(\xi) \right) \Big|_{\xi=\ell}$$

Sum of rational functions. Poles cancel out.

Weight $h(x) = \frac{\langle \ell, x\rangle^m}{m!}$

$$S^L(\mathfrak{p})(\frac{\ell^m}{m!}) = \left(\sum_{s \text{ vertex of } \mathfrak{p}} S^L(s + C_s)_{[m]}(\xi) \right) |_{\xi=\ell}$$

Sum of rational functions. Poles cancel out.

Example $\sum_{n=a}^{b} e^{n\xi} = \frac{e^{a\xi}}{1-e^{\xi}} + \frac{e^{b\xi}}{1-e^{-\xi}} = \frac{e^{a\xi}-e^{(b+1)\xi}}{1-e^{\xi}}$.

$$\sum_{n=a}^{b} \frac{n^m}{m!} = (\frac{e^{a\xi} - e^{(b+1)\xi}}{1 - e^{\xi}})_{[m]}|_{\xi=1}$$

$$= \frac{B_{m+1}(b+1) - B_{m+1}(a)}{(m+1)!}$$

Dilating $\mathfrak{p}$

When $\mathfrak{p}$ is dilated, replaced by $t\mathfrak{p}$,
vertex s is replaced by ts,
cone C_s does not change.

$$S^L(t\mathfrak{p})(\frac{\ell^m}{m!}) = \left(\sum_s S^L(ts + C_s)_{[m]}(\xi)\right)|_{\xi=\ell}$$

[Similar phenomenon for multi-parametric polytopes]

Ehrhart theory $\Leftarrow$ How $S^L(ts + C_s)_{[m]}(\xi)$ depends on $t \in \mathbb{R}$

For a given linear cone C, how does $S^L(u + C)_{[m]}(\xi)$ depend on $u \in \mathbb{R}^d$?

Periodicity

For $v \in \mathbb{Z}^d$,

$$S^L(v + u + C) = \mathrm{e}^{\langle \xi, v \rangle} S^L(u + C)$$

$$M^L(u, C)(\xi) := \mathrm{e}^{-\langle \xi, u \rangle} S^L(u + C)(\xi)$$

is a $\mathbb{Z}^d$-periodic function of u

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$$S^L(u + C)_{[m]}(\xi) = \sum_{j=0}^{m+d} \frac{\langle \xi, u \rangle^{m+d-j}}{(m + d - j)!} M^L(u, C)_{[-d+j]}(\xi)$$

Piecewise polynomial

$$S^L(u+C)_{[m]}(\xi) = \sum_{j=0}^{m+d} \frac{\langle \xi, u\rangle^{m+d-j}}{(m+d-j)!} M^L(u,C)_{[-d+j]}(\xi)$$

$M^L(u,C)_{[-d+j]}(\xi)$ are expressed as step-polynomials of $u \in \mathbb{R}^d$, hence piecewise polynomial function of $u \in \mathbb{R}^d$, with one-sided-continuity on boundary of pieces.

Example.

$M(s+\mathbb{R}_{\geq 0})(\xi) = \frac{\mathrm{e}^{\{-s\}\xi}}{1-\mathrm{e}^{\xi}}$

$M(s+\mathbb{R}_{\geq 0})_{[m]}(\xi) = -\frac{B_{m+1}(\{-s\})}{(m+1)!}\xi^m$

polynomial of degree $m+1$ on pieces $s \in]n, n+1]$, left continuous on each piece.

Periodicity

For $v \in \mathbb{Z}^d$,

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$$M^L(u,C)(\xi) := \mathrm{e}^{-\langle \xi, u\rangle} S^L(u+C)(\xi)$$

is a periodic function of u

$$S^L(t\,s+C)_{[m]}(\xi) = \sum_{j=0}^{m+d} t^{m+d-j} \frac{\langle \xi, s\rangle^{m+d-j}}{(m+d-j)!} M^L(t\,s, C)_{[-d+j]}(\xi)$$

quasi-polynomial of degree $m+d$ with respect to t .

Highest degree terms of Ehrhart quasi-polynomial

when the weight is a power of a linear form

Combining with Brion's formula, get

$$S(t\mathfrak{p})(\frac{\ell^m}{m!}) = \left(\sum_{s \text{ vertex of } \mathfrak{p}} S(ts + C_s)_{[m]}(\xi) \right) \Big|_{\xi=\ell}$$

$$= \left(\sum_{s \text{ vertex of } \mathfrak{p}} \sum_{j=0}^{m+d} t^{m+d-j} \frac{\langle \xi, s \rangle^{m+d-j}}{(m+d-j)!} M(t\, s, C_s)_{[-d+j]}(\xi) \right) \Big|_{\xi=\ell}$$

For computing the highest degree coefficients, of terms $t^{m+d}, t^{m+d-1}, \ldots, t^{m+d-k}$, need only lowest degree components

$$M(t\, s, C_s)_{[-d]},\ M(t\, s, C_s)_{[-d+1]},\ \ldots, M(t\, s, C_s)_{[-d+k]}$$

Example in dim 2.

$$M(s,C)(\xi) := \mathrm{e}^{-\langle \xi, s\rangle} S(s+C)(\xi) = \frac{\mathrm{e}^{\{-s_1\}\xi_1}\,\mathrm{e}^{\{-s_2\}\xi_2}}{(1-\mathrm{e}^{\xi_1})\,(1-\mathrm{e}^{\xi_2})} =$$

$$\frac{1}{\xi_1\xi_2} \qquad \text{degree } (-d) = -2$$

$$+B_1(\{-s_2\})\frac{1}{\xi_1} + B_1(\{-s_1\})\frac{1}{\xi_2} \qquad \text{degree } (-d+1) = -1$$

$$+B_1(\{-s_1\})B_2(\{-s_2\})$$

$$+\frac{B_2(\{-s_2\})}{2}\frac{\xi_2}{\xi_1} + \frac{B_2(\{-s_1\})}{2}\frac{\xi_1}{\xi_2} \qquad \text{degree } (-d+2) = 0$$

$$+\cdots \qquad \text{degree } \geq 1$$

$B_1(u) = u - \frac{1}{2}$, $B_2(u) = u^2 - u + \frac{1}{6}$.

Approximation. Fix $k \leq d$.

$$S(t\,s + C)_{[m]}(\xi) = \sum_{j=0}^{m+d} t^{m+d-j} \frac{\langle \xi, s \rangle^{m+d-j}}{(m+d-j)!} M(t\,s, C)_{[-d+j]}(\xi)$$

For each supporting cone C_s of $\mathfrak{p}$, we construct functions $M^{approx}(u, C_s)(\xi)$ which approximate $M(u, C_s)(\xi)$ in the sense of power series w.r.to ξ,
i.e. have the same lowest degree components as $M(u, C_s)(\xi)$.

$$\begin{aligned} M(\xi) &= M_{[-d]}(\xi) + \cdots + M_{[-d+k]}(\xi) + \text{ higher degree} \\ M^{approx}(\xi) &= M_{[-d]}(\xi) + \cdots + M_{[-d+k]}(\xi) + \text{ higher degree} \end{aligned}$$

Approximations of the generating function of a cone

Use linear combinations of intermediate generating functions of the cone $u + C$

$$M_{\mathcal{L}}^{approx}(u, C)(\xi) = \mathrm{e}^{-\langle \xi, u\rangle} \sum_{L \in \mathcal{L}} \rho_{\mathcal{L}}(L) S^L(u + C)(\xi)$$

$\mathcal{L}$ any family of subspaces which contains all subspaces parallel to the faces of codimension $\leq k$ of C and is closed under sum. $\rho_{\mathcal{L}}(L)$ Moebius coefficients of $\mathcal{L}$.

When C is simplicial, smallest such family is just the set of faces of codimension $\leq k$, (sums of subspaces not needed), Moebius function easy to compute.

- First proof that $M^{approx}_{\mathcal{L}_s}(u, C)(\xi)$ is an approximation: based on Euler-MacLaurin formula for a polyhedral cone (Baldoni, B., Vergne, 2008)
- New proof (2013): based on Poisson summation formula (and one-sided continuity).

Heuristically, $S(u+C)(\xi) = \sum_{x\in\mathbb{Z}^d} \phi(x) = \sum_{\gamma\in\mathbb{Z}^d} \hat{\phi}(2\pi\gamma)$

$\phi(x) := \mathrm{e}^{\langle\xi,x\rangle}[u+C](x)$

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$\hat{\phi}(2\pi\gamma) = \int_{u+C} \mathrm{e}^{\langle \xi + 2i\pi\gamma, x \rangle}\, dx =: I(u+C)(\xi + 2i\pi\gamma)$

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$$S(u + C)(\xi) = \sum_{\gamma\in\mathbb{Z}^d} I(u + C)(\xi + 2i\pi\gamma)$$

Dual lattice to projected lattice: $(\Lambda_{\mathbb{R}^d/L})^* = \mathbb{Z}^d \cap L^\perp$.

$$S^L(u + C)(\xi) = \sum_{\gamma\in\mathbb{Z}^d\cap L^\perp} I(u + C)(\xi + 2i\pi\gamma)$$

Barvinok's linear combination

$\mathcal{L}$ finite family of subspaces L.
$\rho(L) \in \mathbb{Z}$.

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})$$

Fix $k \leq \dim \mathfrak{p}$.

- family $\mathcal{L}$ contains the linear spaces parallel to all faces of codimension $\leq k$ of $\mathfrak{p}$,
- family $\mathcal{L}$ is closed under sum,
  `drawings on blackboard`
- coefficients satisfy Moebius property

$$[\bigcup_{L \in \mathcal{L}} L^{\perp}] = \sum_{L \in \mathcal{L}} \rho(L) [L^{\perp}].$$

$L^{\perp}$ orthogonal of L in dual space of $\mathbb{R}^d$

Two canonical families of slicing subspaces for a polytope $\mathfrak{p}$

In order to use Brion's theorem, need to approximate generating functions of cone C_s, for all vertices s of $\mathfrak{p}$.
Need a family of slicing subspaces $\mathcal{L}_s$ for each vertex s of $\mathfrak{p}$.
Two canonical such families.

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- *Barvinok* (original family of A.Barvinok):
  same family $\mathcal{L}_s = \mathcal{L}$ for all vertices, $\mathcal{L}$ the smallest family which contains all subspaces parallel to the faces of codimension $\leq k$ of $\mathfrak{p}$ and is closed under sum.

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- *ConeByCone*: for a vertex s, $\mathcal{L}_s$ the smallest family which contains all subspaces parallel to the faces of codimension $\leq k$ of C_s and is closed under sum.
  (Easy when $\mathfrak{p}$ is a simple polytope)

Three quasi-polynomials for a polytope $\mathfrak{p}$

- $S(t\mathfrak{p}, h) = \sum_{x \in t\mathfrak{p} \cap \mathbb{Z}^d} h(x)$
- $\mathrm{Barvinok}_{\mathfrak{p},h,k}(t) = \sum_{L \in \mathcal{L}} \rho_{\mathcal{L}}(L) S^L(t\mathfrak{p}, h)$
  where $\mathcal{L}$ family of sums of faces of codim $\leq k$.
- $\mathrm{ConeByCone}_{\mathfrak{p},h,k}(t)$

Three quasi-polynomials for a polytope $\mathfrak{p}$

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  where $\mathcal{L}$ family of sums of faces of codim $\leq k$.
- $\text{ConeByCone}_{\mathfrak{p},h,k}(t)$ What is it ?

The third quasi-polynomial when $\mathfrak{p}$ is simple

Fix k, $0 \leq k \leq d$.

Theorem (B.B.DeL.K.V. 2013). $\mathfrak{p}$ simple polytope.
1) For each vertex s of $\mathfrak{p}$, let $\mathcal{L}_s$ be the family of faces of codimension $\leq k$ of the supporting cone C_s. Then the sum

$$ConeByCone(\mathfrak{p})(\xi) := \sum_s \sum_{L \in \mathcal{L}_s} \rho_{\mathcal{L}_s}(L) S^L(s + C_s)(\xi)$$

is analytic at $\xi = 0$.
2) For $h(x) = \frac{\langle \ell, x\rangle^m}{m!}$, polynomial weight on $\mathbb{R}^d$, define

$$ConeByCone(\mathfrak{p}, h) = ConeByCone(\mathfrak{p})_{[m]}(\xi)\big|_{\xi=\ell}$$

Then $ConeByCone(t\mathfrak{p}, h)$ is a quasi-polynomial with the same $k+1$ highest degree terms as $S(t\mathfrak{p}, h)$.
Proof of 1): poles and residues of $S^L(s + C_s)(\xi)$ are computed using Poisson summation formula.

Step-polynomials, polynomial time algorithms.

ask Jesus, Matthias ...

For $h(x) = \frac{\langle \ell, x\rangle^m}{m!}$
$\mathrm{Barvinok}_{\mathfrak{p},h,k}(t) = \sum_{L\in\mathcal{L}} \rho_{\mathcal{L}}(L) S^L(t\mathfrak{p})_{[m]}(\ell)$
$\mathrm{ConeByCone}_{\mathfrak{p},h,k}(t) = \sum_s \sum_{L\in\mathcal{L}_s} \rho_{\mathcal{L}_s}(L) S^L(s + C_s)_{[m]}(\ell)$

• Read out of $S^L(t\,s + C_s)_{[m]}(\xi)$ how to express $\mathrm{Barvinok}_{\mathfrak{p},h,k}(t)$ and $\mathrm{ConeByCone}_{\mathfrak{p},h,k}(t)$ in terms of a finite family of step-polynomials associated with the input $(\mathfrak{p}, \ell, m)$.

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• Moebius coefficients for a simplicial cone. A face L of codimension j is defined by $d - j$ edges of the cone.
$\rho(L) = (-1)^{k-j}\,\text{binomial}(d - j - 1, d - k - 1)$

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$\rho(L) = (-1)^{k-j}\,\text{binomial}(d - j - 1, d - k - 1)$
• Moebius coefficients for a simplex. A face L of codimension j is defined by $d + 1 - j$ vertices, need to take also sums of subspaces parallel to faces.
$\rightarrow$ Poset of partitions of $\{1, \ldots, d + 1\}$ into blocks of either size 1 (singletons) or size $\geq d + 1 - k$.

THANK YOU