Elementary recursive bounds for Hilbert 17th problem (Part I)

Daniel Perrucci
Universidad de Buenos Aires, Argentina

joint work with

Henri Lombardi
Université de Franche-Comté, France

and

Marie-Françoise Roy
IRMAR CNRS / Université de Rennes 1, France

Polyhedra, Lattices, Algebra, and Moments
IMS, Singapore - January 13, 2014

Hilbert 17th problem

$$P \geq 0 \ \text{ in } \ \mathbb{R}^k \quad \Longrightarrow \quad P = \sum_i G_i^2, \quad G_i \in \mathbb{R}(x_1, \ldots, x_k) \text{ ?}$$

- Artin '27: Affirmative answer. Non-constructive.

- Kreisel '57 - Daykin '61 - Lombardi '90 - Schmid '00: Constructive proofs leading to primitive recursive degree bounds depending on k and $d = \deg P$.

- Our work: Constructive proof leading to an elementary recursive degree bound:

$$2^{2^{2^{d^{4^k}}}}.$$

Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \quad \Longrightarrow \quad P = \sum_i G_i^2, \quad G_i \in \mathbb{R}(x_1, \ldots, x_k) \ ?$$

- Artin '27: Affirmative answer. Non-constructive.

- Kreisel '57 - Daykin '61 - Lombardi '90 - Schmid '00: Constructive proofs leading to primitive recursive degree bounds depending on k and $d = \deg P$.

- Our work: Constructive proof leading to an elementary recursive degree bound:

$$2^{2^{2^{d^{4^k}}}}.$$

Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \quad \Longrightarrow \quad P = \sum_i G_i^2, \quad G_i \in \mathbb{R}(x_1, \ldots, x_k) \text{ ?}$$

- Artin '27: Affirmative answer. Non-constructive.

- Kreisel '57 - Daykin '61 - Lombardi '90 - Schmid '00: Constructive proofs leading to primitive recursive degree bounds depending on k and $d = \deg P$.

- Our work: Constructive proof leading to an elementary recursive degree bound:

$$2^{2^{2^{d^{4^k}}}}.$$

Hilbert 17th problem

$$P \geq 0 \ \text{ in } \ \mathbb{R}^k \quad \Longrightarrow \quad P = \sum_i G_i^2, \quad G_i \in \mathbb{R}(x_1, \ldots, x_k) \ ?$$

- Artin '27: Affirmative answer. Non-constructive.

- Kreisel '57 - Daykin '61 - Lombardi '90 - Schmid '00: Constructive proofs leading to primitive recursive degree bounds depending on k and $d = \deg P$.

- Our work: Constructive proof leading to an elementary recursive degree bound:

$$2^{2^{2^{d^{4^k}}}}.$$

Positivstellensatz (Krivine '64, Stengle '74)

- **K** an ordered field, **R** a real closed extension of **K**,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H}(x) : \begin{cases} P_i(x) \neq 0 & \text{for} \quad i \in I_{\neq} \\ P_i(x) \geq 0 & \text{for} \quad i \in I_{\geq} \\ P_i(x) = 0 & \text{for} \quad i \in I_{=} \end{cases} \quad \text{no solution in } \mathbf{R}^k \quad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2 \Big) \prod_{i \in I} P_i \quad (k_{I,j} > 0),$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x]$$

such that $\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0.$

Positivstellensatz (Krivine '64, Stengle '74)

- **K** an ordered field, **R** a real closed extension of **K**,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H}(x) : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases} \quad \text{no solution in } \mathbf{R}^k \qquad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2\Big) \prod_{i \in I} P_i \quad (k_{I,j} > 0),$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x]$$

such that $\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0.$

Positivstellensatz (Krivine '64, Stengle '74)

- **K** an ordered field, **R** a real closed extension of **K**,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H}(x) : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases} \quad \text{no solution in } \mathbf{R}^k \quad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2 \Big) \prod_{i \in I} P_i \quad (k_{I,j} > 0),$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x]$$

such that
$$\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0.$$

Incompatibilities

$$\mathcal{H}(x): \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

$$\downarrow \mathcal{H} \downarrow : \qquad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

with

$S \in \left\{ \prod_{i \in I_{\neq}} P_i^{2e_i} \right\}$ ← monoid associated to $\mathcal{H}$

$N \in \left\{ \sum_{I \subset I_{\geq}} \left(\sum_j k_{I,j} Q_{I,j}^2 \right) \prod_{i \in I} P_i \right\}$ ← cone associated to $\mathcal{H}$

$Z \in \langle P_i \mid i \in I_{=} \rangle$ ← ideal associated to $\mathcal{H}$

Degree of an incompatibility

$$\mathcal{H}(x): \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

$$\downarrow \mathcal{H} \downarrow : \qquad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2 \Big) \prod_{i \in I} P_i, \qquad Z = \sum_{i \in I_{=}} Q_i P_i$$

the degree of $\mathcal{H}$ is the maximum degree of

$$S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad Q_{I,j}^2 \prod_{i \in I} P_i \;\; (I \subset I_{\geq}, j), \qquad Q_i P_i \;\; (i \in I_{=}).$$

Example :

$$\begin{cases} x & \neq & 0 \\ y - x^2 - 1 & \geq & 0 \\ xy & = & 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow x \neq 0, \quad y - x^2 - 1 \geq 0, \quad xy = 0 \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x^2(y - x^2 - 1) + x^4}_{\geq 0} + \underbrace{(-x^2 y)}_{=0} = 0.$$

The degree of this is incompatibility is 4.

Example :

$$\begin{cases} x & \neq & 0 \\ y - x^2 - 1 & \geq & 0 \\ xy & = & 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow\ x \neq 0,\ \ y - x^2 - 1 \geq 0,\ \ xy = 0\ \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x^2(y - x^2 - 1) + x^4}_{\geq 0} + \underbrace{(-x^2 y)}_{=0} = 0.$$

The degree of this is incompatibility is 4.

Example :

$$\begin{cases} x & \neq & 0 \\ y - x^2 - 1 & \geq & 0 \\ xy & = & 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow\ x \neq 0,\ \ y - x^2 - 1 \geq 0,\ \ xy = 0\ \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x^2(y - x^2 - 1) + x^4}_{\geq 0} + \underbrace{(-x^2 y)}_{=0} = 0.$$

The degree of this is incompatibility is 4.

Example 2:

$$\begin{cases} x > 0 \\ y - x \geq 0 \\ xy \leq 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow\ x > 0,\ \ y - x \geq 0,\ \ xy \leq 0\ \downarrow$ equivalent to

$\downarrow\ x \neq 0,\ \ x \geq 0,\ \ y - x \geq 0,\ \ -xy \geq 0\ \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x(y-x) - xy}_{\geq 0} + \underbrace{0}_{=0} = 0.$$

The degree of this is incompatibility is 2.

Example 2:

$$\begin{cases} x > 0 \\ y - x \geq 0 \\ xy \leq 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow\ x > 0,\ \ y - x \geq 0,\ \ xy \leq 0\ \downarrow$ equivalent to

$\downarrow\ x \neq 0,\ \ x \geq 0,\ \ y - x \geq 0,\ \ -xy \geq 0\ \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x(y-x) - xy}_{\geq 0} + \underbrace{0}_{=0} = 0.$$

The degree of this is incompatibility is 2.

Example 2:

$$\begin{cases} x & > & 0 \\ y-x & \geq & 0 \\ xy & \leq & 0 \end{cases} \quad \text{no solution in } \mathbb{R}^2$$

$\downarrow\ x > 0,\ \ y-x \geq 0,\ \ xy \leq 0\ \downarrow$ equivalent to

$\downarrow\ x \neq 0,\ \ x \geq 0,\ \ y-x \geq 0,\ \ -xy \geq 0\ \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x(y-x) - xy}_{\geq 0} + \underbrace{0}_{=0} = 0.$$

The degree of this is incompatibility is 2.

Positivstellensatz: Constructive proofs

• Lombardi '90, Schmid '00: Primitive recursive degree bounds on k, $d = \max \deg P_i$ and $s = \# P_i$.

Lombardi '90: Based on Hörmander algorithm for quantifier elimination:

- exponential tower of height $k + 4$,
- $d \log(d) + \log \log(s) + c$ on the top.

• Our work: Based on cylindrical algebraic decomposition + sign determination + Hermite theory + Laplace's proof of FTA. Elementary recursive degree bound in k, d and s:

$$2^{2^{2^{\max\{2,d\}^{4^k}} + s^{2^k} \max\{2,d\}^{16^k \mathrm{bit}(d)}}}.$$

Positivstellensatz: Constructive proofs

- Lombardi '90, Schmid '00: Primitive recursive degree bounds on k, $d = \max \deg P_i$ and $s = \# P_i$.

Lombardi '90: Based on Hörmander algorithm for quantifier elimination:

- exponential tower of height $k + 4$,
- $d \log(d) + \log\log(s) + c$ on the top.

- Our work: Based on cylindrical algebraic decomposition + sign determination + Hermite theory + Laplace's proof of FTA. Elementary recursive degree bound in k, d and s:

$$2^{2^{2^{\max\{2,d\}^{4^k}} + s^{2^k} \max\{2,d\}^{16^k \mathrm{bit}(d)}}}.$$

Positivstellensatz: Constructive proofs

• Lombardi '90, Schmid '00: Primitive recursive degree bounds on k, $d = \max \deg P_i$ and $s = \# P_i$.

Lombardi '90: Based on Hörmander algorithm for quantifier elimination:

- exponential tower of height $k + 4$,
- $d \log(d) + \log\log(s) + c$ on the top.

• Our work: Based on cylindrical algebraic decomposition + sign determination + Hermite theory + Laplace's proof of FTA.
Elementary recursive degree bound in k, d and s:

$$2^{2^{2^{\max\{2,d\}^{4^k}} + s^{2^k} \max\{2,d\}^{16^k \mathrm{bit}(d)}}}.$$

Positivstellensatz: Constructive proofs

• Lombardi '90, Schmid '00: Primitive recursive degree bounds on k, $d = \max\deg P_i$ and $s = \#P_i$.

Lombardi '90: Based on Hörmander algorithm for quantifier elimination:

- exponential tower of height $k+4$,
- $d\log(d) + \log\log(s) + c$ on the top.

• Our work: Based on cylindrical algebraic decomposition + sign determination + Hermite theory + Laplace's proof of FTA. Elementary recursive degree bound in k, d and s:

$$2^{2^{2^{\max\{2,d\}^{4^k}} + s^{2^k}\max\{2,d\}^{16^k \mathrm{bit}(d)}}}.$$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff \{ \; P(x) < 0 \quad \text{no solution}$$

$$\iff \begin{cases} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{cases} \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2)P}_{\geq 0} = 0$$

$$\implies P = \frac{(\sum_j R_j^2)P^2}{P^{2e} + \sum_i Q_i^2} = \frac{(\sum_j R_j^2)P^2(P^{2e} + \sum_i Q_i^2)}{(P^{2e} + \sum_i Q_i^2)^2}.$$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff \{\; P(x) < 0 \quad \text{no solution}$$

$$\iff \begin{cases} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{cases} \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2) P}_{\geq 0} = 0$$

$$\implies P = \frac{(\sum_j R_j^2) P^2}{P^{2e} + \sum_i Q_i^2} = \frac{(\sum_j R_j^2) P^2 (P^{2e} + \sum_i Q_i^2)}{(P^{2e} + \sum_i Q_i^2)^2}.$$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff \left\{ P(x) < 0 \quad \text{no solution} \right.$$

$$\iff \left\{ \begin{array}{rcl} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2) P}_{\geq 0} = 0$$

$$\implies P = \frac{(\sum_j R_j^2) P^2}{P^{2e} + \sum_i Q_i^2} = \frac{(\sum_j R_j^2) P^2 (P^{2e} + \sum_i Q_i^2)}{(P^{2e} + \sum_i Q_i^2)^2}.$$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff \{\ P(x) < 0 \quad \text{no solution}$$

$$\iff \begin{cases} P(x) \neq 0 \\ -P(x) \geq 0 \end{cases} \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2)P}_{\geq 0} = 0$$

$$\implies P = \frac{(\sum_j R_j^2)P^2}{P^{2e} + \sum_i Q_i^2} = \frac{(\sum_j R_j^2)P^2(P^{2e} + \sum_i Q_i^2)}{(P^{2e} + \sum_i Q_i^2)^2}.$$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff \left\{ \; P(x) < 0 \quad \text{no solution} \right.$$

$$\iff \left\{ \begin{array}{rcl} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2) P}_{\geq 0} = 0$$

$$\Longrightarrow \quad P = \frac{(\sum_j R_j^2) P^2}{P^{2e} + \sum_i Q_i^2} = \frac{(\sum_j R_j^2) P^2 (P^{2e} + \sum_i Q_i^2)}{(P^{2e} + \sum_i Q_i^2)^2}.$$

Weak inferences: example

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ A + B > 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) + B(x) & > \quad 0 \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow \quad \Leftarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) & > \quad 0 \\ B(x) & \geq \quad 0 \end{cases} \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0.$$

Weak inferences go from right to left.

Weak inferences: example

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ A + B > 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) + B(x) > 0 & \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow \quad \Leftarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) > 0 & \\ B(x) \geq 0 & \end{cases} \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0.$$

Weak inferences go from right to left.

Weak inferences: example

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ \ A + B > 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) + B(x) & > \quad 0 \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ \ A > 0,\ \ B \geq 0 \downarrow \quad \Leftarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) & > \quad 0 \\ B(x) & \geq \quad 0 \end{cases} \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0.$$

Weak inferences go from right to left.

Weak inferences: example

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ \ A+B>0 \downarrow \quad \Rightarrow \quad \left\{ \begin{array}{lll} \mathcal{H}(x) & & \\ A(x)+B(x) & > & 0 \end{array} \right. \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ \ A>0,\ \ B \geq 0 \downarrow \quad \Leftarrow \quad \left\{ \begin{array}{lll} \mathcal{H}(x) & & \\ A(x) & > & 0 \\ B(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0.$$

Weak inferences go from right to left.

Weak inferences: example

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H}, \ \ A + B > 0 \downarrow \quad \Rightarrow \quad \left\{ \begin{array}{lll} \mathcal{H}(x) & & \\ A(x) + B(x) & > & 0 \end{array} \right. \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H}, \ \ A > 0, \ \ B \geq 0 \downarrow \quad \Leftarrow \quad \left\{ \begin{array}{lll} \mathcal{H}(x) & & \\ A(x) & > & 0 \\ B(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0.$$

Weak inferences go from right to left.

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$\downarrow \mathcal{H},\ A+B>0 \downarrow$ equiv. to $\downarrow \mathcal{H},\ A+B \neq 0,\ A+B \geq 0 \downarrow$

$$\underbrace{(A+B)^{2e}S}_{>0} + \underbrace{N + N'(A+B)}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{\sum_{i=0}^{2e-1}\binom{2e}{i}A^iB^{2e-1}S + N + N'A + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0,\ A \geq 0,\ B \geq 0 \downarrow$ equiv. to $\downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$$\downarrow \mathcal{H},\ A + B > 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A + B \neq 0,\ A + B \geq 0 \downarrow$$

$$\underbrace{(A+B)^{2e}S}_{>0} + \underbrace{N + N'(A+B)}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{\sum_{i=0}^{2e-1} \binom{2e}{i} A^i B^{2e-i} S + N + N'A + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow \mathcal{H},\ A \neq 0,\ A \geq 0,\ B \geq 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$$\downarrow \mathcal{H},\ A+B>0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A+B \neq 0,\ A+B \geq 0 \downarrow$$

$$\underbrace{(A+B)^{2e}S}_{>0} + \underbrace{N + N'(A+B)}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{\sum_{i=0}^{2e-1} \binom{2e}{i} A^i B^{2e-1} S + N + N'A + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow \mathcal{H},\ A \neq 0,\ A \geq 0,\ B \geq 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$$\downarrow \mathcal{H},\ A + B > 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A + B \neq 0,\ A + B \geq 0 \downarrow$$

$$\underbrace{(A+B)^{2e} S}_{>0} + \underbrace{N + N'(A+B)}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{A^{2e} S}_{>0} + \underbrace{\sum_{i=0}^{2e-1} \binom{2e}{i} A^i B^{2e-1} S + N + N'A + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow \mathcal{H},\ A \neq 0,\ A \geq 0,\ B \geq 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow$$

$$A > 0, \quad B \geq 0 \qquad \vdash \qquad A + B > 0$$

$$\downarrow \mathcal{H},\ A+B > 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A+B \neq 0,\ A+B \geq 0 \downarrow$$

$$\underbrace{(A+B)^{2e}S}_{>0} \ + \ \underbrace{N + N'(A+B)}_{\geq 0} \ + \ \underbrace{Z}_{=0} \ = \ 0$$

$$\underbrace{A^{2e}S}_{>0} \ + \ \underbrace{\sum_{i=0}^{2e-1} \binom{2e}{i} A^i B^{2e-1} S + N + N'A + N'B}_{\geq 0} \ + \ \underbrace{Z}_{=0} \ = \ 0$$

$$\downarrow \mathcal{H},\ A \neq 0,\ A \geq 0,\ B \geq 0 \downarrow \quad \text{equiv. to} \quad \downarrow \mathcal{H},\ A > 0,\ B \geq 0 \downarrow$$

$$A > 0, \quad B \geq 0 \qquad \vdash \qquad A + B > 0$$

What about degrees?

$$\underline{(A+B)^{2e}S} + \underline{N} + \underline{N'(A+B)} + \underline{Z} = 0$$

$$\underline{A^{2e}S} + \sum_{i=0}^{2e-1} \underline{\binom{2e}{i} A^i B^{2e-1} S} + \underline{N} + \underline{N'A} + \underline{N'B} + \underline{Z} = 0$$

initial incompatibility
degree
$\leq \delta$

final incompatibility
degree
$\leq \delta + \max\{1, 2e\} \cdot$
$(\max\{\deg A, \deg B\} - \deg\{A + B\})$

$$A > 0, \quad B \geq 0 \qquad \vdash \qquad A + B > 0$$

What about degrees?

$$\underline{(A+B)^{2e}S} + \underline{N} + \underline{N'(A+B)} + \underline{Z} = 0$$

$$\underline{A^{2e}S} + \sum_{i=0}^{2e-1} \underline{\binom{2e}{i} A^i B^{2e-1} S} + \underline{N} + \underline{N'A} + \underline{N'B} + \underline{Z} = 0$$

initial incompatibility
degree
$\leq \delta$

final incompatibility
degree
$\leq \delta + \max\{1, 2e\} \cdot$
$(\max\{\deg A, \deg B\} - \deg\{A + B\})$

$$A \neq 0 \quad \Longrightarrow \quad A < 0 \quad \vee \quad A > 0$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ A < 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) < 0 \end{cases} \quad \text{no solution}$$

$$\downarrow \mathcal{H},\ A > 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) > 0 \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ A \neq 0 \downarrow \quad \Leftarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) \neq 0 \end{cases} \quad \text{no solution}$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

Again, from right to left.

$$A \neq 0 \quad \Longrightarrow \quad A < 0 \quad \vee \quad A > 0$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ \ A < 0 \downarrow \ \Rightarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & < \ 0 \end{cases} \quad \text{no solution}$$

$$\downarrow \mathcal{H},\ \ A > 0 \downarrow \ \Rightarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & > \ 0 \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ \ A \neq 0 \downarrow \ \Leftarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & \neq \ 0 \end{cases} \quad \text{no solution}$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

Again, from right to left.

$$A \neq 0 \quad \Longrightarrow \quad A < 0 \quad \vee \quad A > 0$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ \ A < 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) & < \quad 0 \end{cases} \quad \text{no solution}$$

$$\downarrow \mathcal{H},\ \ A > 0 \downarrow \quad \Rightarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) & > \quad 0 \end{cases} \quad \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ \ A \neq 0 \downarrow \quad \Leftarrow \quad \begin{cases} \mathcal{H}(x) & \\ A(x) & \neq \quad 0 \end{cases} \quad \text{no solution}$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

Again, from right to left.

$$A \neq 0 \quad \Longrightarrow \quad A < 0 \quad \vee \quad A > 0$$

Let $\mathcal{H}$ be any system of sign conditions.

$$\downarrow \mathcal{H},\ \ A < 0 \downarrow \ \Rightarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & < \ 0 \end{cases} \ \text{no solution}$$

$$\downarrow \mathcal{H},\ \ A > 0 \downarrow \ \Rightarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & > \ 0 \end{cases} \ \text{no solution}$$

$$\Downarrow$$

$$\downarrow \mathcal{H},\ \ A \neq 0 \downarrow \ \Leftarrow \ \begin{cases} \mathcal{H}(x) & \\ A(x) & \neq \ 0 \end{cases} \ \text{no solution}$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

Again, from right to left.

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$

$\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$ $\qquad$ $\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$

$\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0$$

$$\underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A$$

$$A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$ $\qquad$ $\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$ $\qquad$ $\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$

$\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$\downarrow \mathcal{H},\ A < 0 \downarrow$ degree $\leq \delta_1$

$\downarrow \mathcal{H},\ A > 0 \downarrow$ degree $\leq \delta_2$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A$$

$$\underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

multiplying these we obtain:

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$\downarrow \mathcal{H},\ A \neq 0 \downarrow$ degree $\leq \delta_1 + \delta_2$

$$\dfrac{A \neq 0 \implies \begin{array}{c} A = 0 \;\vee\; A \neq 0 \\ A < 0 \;\vee\; A > 0 \end{array}}{A = 0 \;\vee\; A < 0 \;\vee\; A > 0}$$

Similarly,

$$\dfrac{\begin{array}{rcl} & \vdash & A = 0 \;\vee\; A \neq 0 \\ A \neq 0 & \vdash & A < 0 \;\vee\; A > 0 \end{array}}{\vdash \; A = 0 \;\vee\; A < 0 \;\vee\; A > 0}$$

$$\begin{array}{lcl} & \vdash & A = 0 \quad \vee \quad A \neq 0 \\ A \neq 0 & \vdash & A < 0 \quad \vee \quad A > 0 \\ \hline & \vdash & A = 0 \quad \vee \quad A < 0 \quad \vee \quad A > 0 \end{array}$$

as follows:

$$\downarrow \mathcal{H},\ A = 0 \downarrow \qquad \downarrow \mathcal{H},\ A < 0 \downarrow \qquad \downarrow \mathcal{H},\ A > 0 \downarrow$$

$$\downarrow \qquad\qquad \searrow \qquad \swarrow$$

$$\downarrow \mathcal{H},\ A = 0 \downarrow \qquad\qquad \downarrow \mathcal{H},\ A \neq 0 \downarrow$$

$$\searrow \qquad \swarrow$$

$$\downarrow \mathcal{H} \downarrow$$

Weak Existence

$$\exists t \quad | \quad At = 1 \quad \vdash \quad A \neq 0$$

$$\downarrow \mathcal{H}, \quad A \neq 0 \downarrow$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{S}_{>0} + \underbrace{t^{2e}N}_{\geq 0} + \underbrace{t^{2e}Z + ((At)^{2e} - 1)S}_{=0} = 0$$

$\downarrow \mathcal{H}, \quad At = 1 \downarrow$ involves the new variable t

Weak Existence

$$\exists t \quad | \quad At = 1 \quad \vdash \quad A \neq 0$$

$$\downarrow \mathcal{H}, \;\; A \neq 0 \downarrow$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{S}_{>0} + \underbrace{t^{2e}N}_{\geq 0} + \underbrace{t^{2e}Z + ((At)^{2e} - 1)S}_{=0} = 0$$

$\downarrow \mathcal{H}, \;\; At = 1 \downarrow$ involves the new variable t

Weak Existence

$$\exists t \quad | \quad At = 1 \quad \vdash \quad A \neq 0$$

$$\downarrow \mathcal{H}, \ \ A \neq 0 \ \downarrow$$

$$\underbrace{A^{2e} S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\underbrace{S}_{>0} + \underbrace{t^{2e} N}_{\geq 0} + \underbrace{t^{2e} Z + ((At)^{2e} - 1) S}_{=0} = 0$$

$\downarrow \mathcal{H}, \ \ At = 1 \ \downarrow$ involves the new variable t

$$A \neq 0 \quad \vdash \quad \exists t \quad | \quad At = 1$$

$\downarrow \mathcal{H}, \ At = 1 \downarrow$ involves variable t

$$\underbrace{S}_{>0} + \underbrace{\sum \left(\sum k_{I,j} Q_{I,j}^2(t)\right) \prod P_i}_{\geq 0} + \underbrace{\sum W_j(t) P_j \ + \ W(t)(At-1)}_{=0} = 0$$

$$\underbrace{A^{2e} S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$\downarrow \mathcal{H}, \ A \neq 0 \downarrow$ variable t is eliminated

$$A \neq 0 \quad \vdash \quad \exists t \quad | \quad At = 1$$

$\downarrow\ \mathcal{H},\ \ At = 1\ \downarrow$ involves variable t

$$\underbrace{S}_{>0} + \underbrace{\sum \Big(\sum k_{l,j} Q_{l,j}^2(t) \Big) \prod P_i}_{\geq 0} + \underbrace{\sum W_j(t) P_j \ + \ W(t)(At-1)}_{=0} = 0$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$\downarrow\ \mathcal{H},\ \ A \neq 0\ \downarrow$ variable t is eliminated

$$A \neq 0 \quad \vdash \quad \exists t \quad | \quad At = 1$$

$\downarrow \mathcal{H}, \quad At = 1 \downarrow$ involves variable t

$$\underbrace{S}_{>0} + \underbrace{\sum \left(\sum k_{l,j} Q_{l,j}^2(t) \right) \prod P_i}_{\geq 0} + \underbrace{\sum W_j(t) P_j \; + \; W(t)(At-1)}_{=0} = 0$$

$$\underbrace{A^{2e} S}_{>0} \; + \; \underbrace{N}_{\geq 0} \; + \; \underbrace{Z}_{=0} \; = \; 0$$

$\downarrow \mathcal{H}, \quad A \neq 0 \downarrow$ variable t is eliminated

Simple weak inferences are combined to obtain more interesting weak inferences (Marie-Françoise's talk.)

Example

We want an incompabitility

$$\downarrow\ y - x \neq 0,\ y^3 - x^3 = 0\ \downarrow$$

$$(y - x)^{2e} + N + W(y^3 - x^3) = 0.$$

Example

We want an incompabitility

$$\downarrow\ y - x \neq 0,\ y^3 - x^3 = 0\ \downarrow$$

$$(y - x)^{2e}\ +\ N\ +\ W(y^3 - x^3)\ =\ 0.$$

Thank you for your attention!