Gabor Frames in Finite Dimensions

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Cyclic shift operator

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Modulation operator ($\omega = \exp(2\pi i/N)$)

$$M(x_0, x_1, \ldots, x_{N-1}) = (x_0, \omega x_1, \ldots, \omega^{N-1} x_{N-1})$$

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Definition

A Gabor frame with window $\varphi \in \mathbb{C}^N$ is the set of all time-frequency translates of φ:

$$M^{\lambda} T^{\kappa} \varphi, \ 0 \leq \kappa, \lambda \leq N-1,$$

also called a Weyl-Heisenberg orbit.

For any $\Lambda \subseteq \mathbb{Z}_N^2$ we denote

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Problem (Lawrence, Pfander, Walnut, 2005)

Are there $\varphi \in \mathbb{C}^N$ such that $(\varphi, \mathbb{Z}_N^2)$ is in general position, i. e. (φ, Λ) is linearly independent for all $\Lambda \subseteq \mathbb{Z}_N^2$ with $|\Lambda| = N$?

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Applications:

- Signal recovery
- Compressed sensing
- Operator sampling
- Operator identification

$(\varphi, \mathbb{Z}_N^2) = \{\varphi_k\}$ is an *equal norm tight frame*: we have $\|\varphi_k\| = \|\varphi\|$ and

$$\sum_k |\langle f, \varphi_k\rangle|^2 = N^2 \|\varphi\|^2 \|f\|^2.$$

If $(\varphi, \mathbb{Z}_N^2)$ is in general position, then it is also *maximally robust to erasures*: for any $K \subseteq \mathbb{Z}_N^2$ with $|K| \geq N$ we can reconstruct f from $\{\langle f, \varphi_k\rangle\}_{k\in K}$.

$$f = \sum_{k\in K} \langle f, \varphi_k\rangle \tilde{\varphi}_k,$$

where $\{\tilde{\varphi}_k\}_{k\in K}$ is a dual frame of $\{\varphi_k\}_{k\in K}$. The only previously known equal norm tight frames maximally robust to erasures were the harmonic frames.

Definition

$\mathcal{H} \subseteq \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$ is identifiable with identifier $\varphi \in \mathbb{C}^N$, if the map $H \mapsto H\varphi$ is injective.

Let $\mathcal{H}_\Lambda$ denote the space of operators spanned by $M^\lambda T^\kappa$ for $(\kappa, \lambda) \in \Lambda$. If $(\varphi, \mathbb{Z}_N^2)$ is in general position, then φ is an identifier of $\mathcal{H}_\Lambda$ for every Λ with $|\Lambda| \leq N$.

It is the discrete version of the HRT conjecture.

Conjecture (Heil, Ramanathan, Topiwala, 1996)

For any nonzero $f \in L^2(\mathbb{R})$*, every finite set of time-frequency translates of* f *is linearly independent.*

SIC-POVM

Problem

Is there $\varphi \in \mathbb{C}^N$ such that the vectors in $(\varphi, \mathbb{Z}_N^2)$ are pairwise equiangular?

The Clifford group is the normalizer of the WH group in $\mathrm{GL}_2(\mathbb{C})$.

$$\text{Clifford group}/\text{WH group} \cong \mathrm{SL}_2(\mathbb{Z}_N).$$

There is a Clifford unitary, $U_{\mathcal{Z}}$, with order 3 (up to phase) called the *Zauner* matrix.

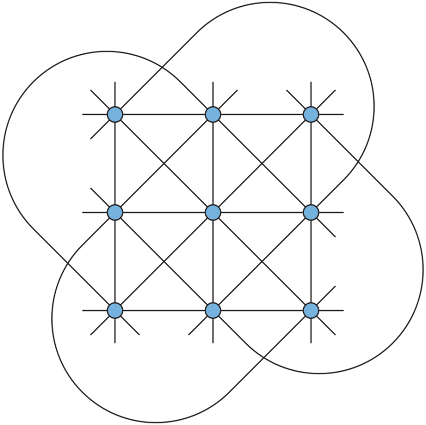
Conjecture (Zauner, 1999)

For some eigenvector of $U_{\mathcal{Z}}$, say φ (called the fiducial vector), the the vectors in $(\varphi, \mathbb{Z}_N^2)$ are pairwise equiangular.

Verified for all $N \leq 67$. Applications:

- Quantum computing, tomography, cryptography.
- Spherical designs.
- Signal processing.

Hughston's observation: SIC-POVMs in $\mathbb{C}^3$ form the Hesse



configuration.

Dang, Blanchfield, Bengtsson, Appleby, tried to find similar conifgurations for $N > 3$.
In general, they studied the WH-orbits of eigenvectors of Clifford unitaries (not necessarily $U_{\mathcal{Z}}$), when N is odd, square-free. Their results can be generalized to all odd N.

Let $\Lambda \subseteq \mathbb{Z}_N^2$ with $|\Lambda| = N$. The column vectors $M^\lambda T^\kappa z$ form a matrix, denoted by D_Λ, where

$$z = (z_0, z_1, \ldots, z_{N-1}) \in \mathbb{C}^N$$

is a variable vector. Define

$$P_\Lambda(z) = \det(D_\Lambda).$$

P_Λ is a homogeneous polynomial in N complex variables, of degree N. The set of zeroes of P_Λ is either the entire space $\mathbb{C}^N$ or has measure zero.

Let $\Lambda = \{(0,0),(0,2),(0,3),(4,1),(4,5),(5,0)\} \subseteq \mathbb{Z}_6^2$. The columns are $z, M^2z, M^3z, MT^4z, M^5T^4z, T^5z$

$$D_\Lambda = \left(\begin{array}{ccc|cc|c} z_0 & z_0 & z_0 & z_2 & z_2 & z_1 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 & \omega z_3 & \omega^5 z_3 & z_2 \\ z_2 & \omega^4 z_2 & z_2 & \omega^2 z_4 & \omega^4 z_4 & z_3 \\ z_3 & z_3 & \omega^3 z_3 & \omega^3 z_5 & \omega^3 z_5 & z_4 \\ z_4 & \omega^2 z_4 & z_4 & \omega^4 z_0 & \omega^2 z_0 & z_5 \\ \underbrace{z_5 \quad \omega^4 z_5 \quad \omega^3 z_5}_{D_0} & & & \underbrace{\omega^5 z_1 \quad \omega z_1}_{D_4} & & \underbrace{z_0}_{D_5} \end{array}\right)$$

D_κ is the $N \times l_\kappa$ submatrix of D_Λ, whose columns have the form $M^\lambda T^\kappa z$. Here,

$$l_0 = 3, l_1 = l_2 = l_3 = 0, l_4 = 2, l_5 = 1$$

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Let B_0, B_4, B_5 be a partition of $\mathbb{Z}_6$ with $|B_\kappa| = l_\kappa$, and form the $l_\kappa \times l_\kappa$ submatrix of D_κ (say $D_\kappa(B_\kappa)$) whose rows belong to B_κ.

Partition: $B_0 = \{0, 1, 2\}, B_4 = \{3, 4\}, B_5 = \{5\}$.

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\cline{1-3}
\multicolumn{1}{|c}{z_0} & z_0 & \multicolumn{1}{c|}{z_0} & z_2 & z_2 & z_1 \\
\multicolumn{1}{|c}{z_1} & \omega^2 z_1 & \multicolumn{1}{c|}{\omega^3 z_1} & \omega z_3 & \omega^5 z_3 & z_2 \\
\multicolumn{1}{|c}{z_2} & \omega^4 z_2 & \multicolumn{1}{c|}{z_2} & \omega^2 z_4 & \omega^4 z_4 & z_3 \\
\cline{1-5}
z_3 & z_3 & \omega^3 z_3 & \multicolumn{1}{|c}{\omega^3 z_5} & \multicolumn{1}{c|}{\omega^3 z_5} & z_4 \\
z_4 & \omega^2 z_4 & z_4 & \multicolumn{1}{|c}{\omega^4 z_0} & \multicolumn{1}{c|}{\omega^2 z_0} & z_5 \\
\cline{4-6}
z_5 & \omega^4 z_5 & \omega^3 z_5 & \omega^5 z_1 & \omega z_1 & \multicolumn{1}{|c|}{z_0} \\
\cline{6-6}
\multicolumn{3}{c}{\underbrace{\hphantom{z_0 \quad \omega^2 z_1 \quad \omega^3 z_1}}_{D_0}} & \multicolumn{2}{c}{\underbrace{\hphantom{\omega^3 z_5 \quad \omega^3 z_5}}_{D_4}} & \underbrace{\hphantom{z_0}}_{D_5}
\end{array}\right)$$

Every diagonal in $D_0(B_0) \cup D_4(B_4) \cup D_5(B_5)$ gives the same monomial: here, it is $z_0^3 z_1 z_2 z_5$.

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Every diagonal in $D_0(B_0) \cup D_4(B_4) \cup D_5(B_5)$ gives the same monomial: here, it is $z_0^3 z_1 z_2 z_5$.

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$$\text{Partition: } B_0 = \{0, 1, 2\}, B_4 = \{3, 4\}, B_5 = \{5\}.$$

$$D_\Lambda = \left(\begin{array}{ccc|cc|c} \color{red}{z_0} & z_0 & z_0 & z_2 & z_2 & z_1 \\ z_1 & \color{red}{\omega^2 z_1} & \omega^3 z_1 & \omega z_3 & \omega^5 z_3 & z_2 \\ z_2 & \omega^4 z_2 & \color{red}{z_2} & \omega^2 z_4 & \omega^4 z_4 & z_3 \\ \hline z_3 & z_3 & \omega^3 z_3 & \color{red}{\omega^3 z_5} & \omega^3 z_5 & z_4 \\ z_4 & \omega^2 z_4 & z_4 & \omega^4 z_0 & \color{red}{\omega^2 z_0} & z_5 \\ \hline z_5 & \omega^4 z_5 & \omega^3 z_5 & \omega^5 z_1 & \omega z_1 & \boxed{\color{red}{z_0}} \end{array}\right)$$

$$\underbrace{\hphantom{z_0 \quad \omega^2 z_1 \quad \omega^3 z_1}}_{D_0} \quad \underbrace{\hphantom{\omega^2 z_4 \quad \omega^4 z_4}}_{D_4} \quad \underbrace{\hphantom{z_0}}_{D_5}$$

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Every diagonal in $D_0(B_0) \cup D_4(B_4) \cup D_5(B_5)$ gives the same monomial: here, it is $z_0^3 z_1 z_2 z_5$.

Put $A_0 = \{0, 1, 2\}, A_4 = \{3, 4\}, A_5 = \{5\}$. Every partition B_0, B_4, B_5 of $\mathbb{Z}_6$ with $|B_\kappa| = l_\kappa$ is obtained from a permutation $\sigma \in S_N$, that is,

$$\sigma(A_\kappa) = B_\kappa.$$

Let Γ be the subgroup of permutations that leave all A_κ invariant; we will call such permutations *trivial* (with respect to Λ). If $\tau \in \Gamma$, then

$$\sigma(A_\kappa) = \sigma(\tau(A_\kappa)),$$

so σ and $\sigma\tau$ give the same permutation, hence the same monomial.

We have established that

$$\begin{aligned} S_N/\Gamma &\longrightarrow \text{monomials} \\ \sigma &\longmapsto Z^\sigma \end{aligned}$$

so

$$P_\Lambda(z) = \det(D) = \sum_{\sigma \in S_N/\Gamma} c_\sigma Z^\sigma$$

with

$$c_\sigma Z^\sigma = \pm \prod_{\kappa=0}^{N-1} \det(D_\kappa(\sigma(A_\kappa)))$$

c_σ is a product of minors of the $N \times N$ *Fourier matrix*.
We say that the monomial Z is obtained uniquely if there is a unique $\sigma \in S_N/\Gamma$ such that $Z = Z^\sigma$. If so, the coefficient of Z^σ in P_Λ is a product of Fourier minors.

$\sigma = \iota$ (identity).

$$\sigma(A_0) = \{0,1,2\}, \sigma(A_4) = \{3,4\}, \sigma(A_5) = \{5\}$$

$$D_\Lambda = \left(\begin{matrix} \underbrace{\begin{matrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \\ z_3 & z_3 & \omega^3 z_3 \\ z_4 & \omega^2 z_4 & z_4 \\ z_5 & \omega^4 z_5 & \omega^3 z_5 \end{matrix}}_{D_0} & \underbrace{\begin{matrix} z_2 & z_2 \\ \omega z_3 & \omega^5 z_3 \\ \omega^2 z_4 & \omega^4 z_4 \\ \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \\ \omega^5 z_1 & \omega z_1 \end{matrix}}_{D_4} & \underbrace{\begin{matrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ \boxed{z_0} \end{matrix}}_{D_5} \end{matrix}\right)$$

The indices that appear are

$$\sigma(A_0) - 0 = \{0,1,2\}, \sigma(A_4) - 4 = \{5,0\}, \sigma(A_5) - 5 = \{0\}.$$

The monomial is

$$Z^\sigma = Z^\iota = z_0^3 z_1 z_2 z_5.$$

For $\sigma = \iota$ we have

$$c_\iota Z^\iota = \begin{vmatrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \end{vmatrix} \cdot \begin{vmatrix} \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \end{vmatrix} \cdot |z_0|,$$

so

$$c_\iota = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^3 \\ 1 & \omega^4 & 1 \end{vmatrix} \cdot \begin{vmatrix} \omega^3 & \omega^3 \\ \omega^4 & \omega^2 \end{vmatrix} \cdot |1| \neq 0$$

$\sigma = (254)$.

$$\sigma(A_0) = \{0,1,5\}, \sigma(A_4) = \{2,3\}, \sigma(A_5) = \{4\}$$

$$D_\Lambda = \begin{pmatrix} \underbrace{\begin{matrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \\ z_3 & z_3 & \omega^3 z_3 \\ z_4 & \omega^2 z_4 & z_4 \\ z_5 & \omega^4 z_5 & \omega^3 z_5 \end{matrix}}_{D_0} & \underbrace{\begin{matrix} z_2 & z_2 \\ \omega z_3 & \omega^5 z_3 \\ \omega^2 z_4 & \omega^4 z_4 \\ \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \\ \omega^5 z_1 & \omega z_1 \end{matrix}}_{D_4} & \underbrace{\begin{matrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \boxed{z_5} \\ z_0 \end{matrix}}_{D_5} \end{pmatrix}$$

The indices that appear are

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The monomial is

$$Z^\sigma = Z^{(254)} = z_0 z_1 z_4 z_5^3.$$

For $\sigma = (254)$ we have

$$c_{(254)} Z^{(254)} = \begin{vmatrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_5 & \omega^4 z_5 & \omega^3 z_5 \end{vmatrix} \cdot \begin{vmatrix} \omega^2 z_4 & \omega^4 z_4 \\ \omega^3 z_5 & \omega^3 z_5 \end{vmatrix} \cdot |z_5|,$$

so

$$c_{(254)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^3 \\ 1 & \omega^4 & \omega^3 \end{vmatrix} \cdot \begin{vmatrix} \omega^2 & \omega^4 \\ \omega^3 & \omega^3 \end{vmatrix} \cdot |1| \neq 0$$

Are there any monomials obtained uniquely? *Yes*, the *lowest index monomial* (LIM) as proven by Lawrence, Pfander, and Walnut.

1. Pick an element of D_Λ with the lowest index.
2. Erase column and row of this element.
3. Repeat with the remaining submatrix.
4. Multiply all variables.

No matter how you proceed with the above algorithm, you always get the same monomial; furthermore, LPW proved that you can obtain this monomial in

$$|\Gamma| = l_0! l_1! \cdots l_{N-1}!$$

ways, or equivalently, it is obtained uniquely.

$$\left(\begin{array}{cc|c|c} z_0 & z_0 & z_1 & z_3 \\ z_1 & -z_1 & iz_2 & -\ iz_0 \\ z_2 & z_2 & -z_3 & -z_1 \\ z_3 & -z_3 & -\ iz_0 & iz_2 \end{array}\right)$$

$$\left(\begin{array}{cc|c|c} z_0 & z_0 & z_1 & z_3 \\ z_1 & -z_1 & iz_2 & -\ iz_0 \\ z_2 & z_2 & -z_3 & -z_1 \\ z_3 & -z_3 & -\ iz_0 & iz_2 \end{array}\right)$$

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$$\left(\begin{array}{cc|c|c} z_0 & z_0 & z_1 & z_3 \\ z_1 & -z_1 & iz_2 & -\ iz_0 \\ z_2 & z_2 & -z_3 & -z_1 \\ z_3 & -z_3 & -\ iz_0 & iz_2 \end{array}\right)$$

$$\left(\begin{array}{cc|c|c} z_0 & z_0 & z_1 & z_3 \\ z_1 & -z_1 & iz_2 & -\,iz_0 \\ z_2 & z_2 & -z_3 & -z_1 \\ z_3 & -z_3 & -\,iz_0 & iz_2 \end{array}\right)$$

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$l_0!l_1!l_2!l_3! = 2$ diagonals give $z_0^3 z_2$. The coefficient is

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot |-i| \cdot |-i| = 0.$$

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$$D_\Lambda = \left(\begin{array}{cccccc}
\cline{1-3}
\multicolumn{1}{|c}{z_0} & z_0 & \multicolumn{1}{c|}{z_0} & z_2 & z_2 & z_1 \\
\multicolumn{1}{|c}{z_1} & \omega^2 z_1 & \multicolumn{1}{c|}{\omega^3 z_1} & \omega z_3 & \omega^5 z_3 & z_2 \\
\multicolumn{1}{|c}{z_2} & \omega^4 z_2 & \multicolumn{1}{c|}{z_2} & \omega^2 z_4 & \omega^4 z_4 & z_3 \\
\cline{1-3}
z_3 & z_3 & \omega^3 z_3 & \omega^3 z_5 & \omega^3 z_5 & z_4 \\
\cline{4-5}
z_4 & \omega^2 z_4 & z_4 & \multicolumn{1}{|c}{\omega^4 z_0} & \multicolumn{1}{c|}{\omega^2 z_0} & z_5 \\
\cline{4-6}
z_5 & \omega^4 z_5 & \omega^3 z_5 & \omega^5 z_1 & \omega z_1 & \multicolumn{1}{|c|}{z_0} \\
\cline{6-6}
\multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{D_0}} & \multicolumn{2}{c}{\underbrace{\qquad\qquad\qquad}_{D_4}} & \underbrace{\quad}_{D_5}
\end{array}\right)$$

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Theorem (Chebotarev, 1926)

If N is prime, all minors of the $N \times N$ Fourier matrix are nonzero.

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If N is prime, for almost all $\varphi \in \mathbb{C}^N$, the Gabor frame with window φ is in general position.

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Theorem (Equivalent to Chebotarev's)

Let $f : \mathbb{Z}_p \to \mathbb{C}$ be nonzero. Then

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N composite? Focus on the *consecutive index monomial* (CIM). It is the monomial that corresponds to the identity permutation (or any trivial permutation), and is denoted simply by Z.
A first observation is that if the CIM is obtained uniquely, then its coefficient is a product of Vandermonde determinants (up to phase), which are nonzero.

$$D_\Lambda = \left(\begin{array}{ccc|cc|c}
\cline{1-3}
\multicolumn{1}{|c}{z_0} & z_0 & z_0 & z_2 & z_2 & z_1 \\
\multicolumn{1}{|c}{z_1} & \omega^2 z_1 & \omega^3 z_1 & \omega z_3 & \omega^5 z_3 & z_2 \\
\multicolumn{1}{|c}{z_2} & \omega^4 z_2 & z_2 & \omega^2 z_4 & \omega^4 z_4 & z_3 \\
\cline{1-5}
z_3 & z_3 & \omega^3 z_3 & \omega^3 z_5 & \omega^3 z_5 & z_4 \\
z_4 & \omega^2 z_4 & z_4 & \omega^4 z_0 & \omega^2 z_0 & z_5 \\
\cline{4-6}
z_5 & \omega^4 z_5 & \omega^3 z_5 & \omega^5 z_1 & \omega z_1 & \multicolumn{1}{|c|}{z_0} \\
\cline{6-6}
\multicolumn{3}{c}{\underbrace{\hphantom{z_5 \quad \omega^4 z_5 \quad \omega^3 z_5}}_{D_0}} & \multicolumn{2}{c}{\underbrace{\hphantom{\omega^5 z_1 \quad \omega z_1}}_{D_4}} & \multicolumn{1}{c}{\underbrace{\hphantom{z_0}}_{D_5}}
\end{array}\right)$$

$$c_\iota = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^3 \\ 1 & \omega^4 & 1 \end{vmatrix} \cdot \begin{vmatrix} \omega^3 & \omega^3 \\ \omega^4 & \omega^2 \end{vmatrix} \cdot |1| \neq 0$$

Associate Z^{σ} to the random variable X_{σ}, as follows: if

$$Z^{\sigma} = z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_{N-1}^{\alpha_{N-1}}$$

define

$$P[X_{\sigma} = i] = \frac{\alpha_i}{N}$$

We say that X_{σ} is obtained uniquely if and only if

$$X_{\sigma} = X_{\sigma\tau} \Leftrightarrow \tau \in \Gamma$$

We have seen that the indices that appear in Z^{σ} come from the sets $\sigma(A_{\kappa}) - \kappa$ *counting multiplicities*. If some $A_{\kappa} - \kappa$ contains negative numbers, then we change Λ to some Λ' such that all $A_{\kappa} - \kappa$ contain nonnegative numbers

Theorem

For any $S \in \mathrm{Aff}(\mathbb{Z}_N^2)$, *we have*

$$P_\Lambda \neq 0 \Leftrightarrow P_{S(\Lambda)} \neq 0.$$

S satisfies $Sx = Ax + b$, where $A \in \mathrm{GL}_2(\mathbb{Z}_N)$, $b \in \mathbb{Z}_N^2$. When

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \ \gcd(a, N) = 1, b = 0,$$

it follows from the action of $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ on the WH group. When $A \in \mathrm{SL}_2(\mathbb{Z}_N)$, $b = 0$, it follows from the action of the Clifford group. When $A = I$, it follows from $MT = \omega TM$.

Theorem

If $\Lambda' = \Lambda + b$, *then*

$$P_{\Lambda'} = cP_\Lambda,$$

for some nonzero c.

$$\Lambda = \{(0,0),(0,2),(0,3),(4,1),(4,5),(5,0)\}.$$

$$A_0 = \{0,1,2\}, A_4 = \{3,4\}, A_5 = \{5\}.$$

$$D_\Lambda = \left(\underbrace{\begin{matrix} \boxed{\begin{matrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \end{matrix}} \\ \begin{matrix} z_3 & z_3 & \omega^3 z_3 \\ z_4 & \omega^2 z_4 & z_4 \\ z_5 & \omega^4 z_5 & \omega^3 z_5 \end{matrix} \end{matrix}}_{D_0} \underbrace{\begin{matrix} \begin{matrix} z_2 & z_2 \\ \omega z_3 & \omega^5 z_3 \\ \omega^2 z_4 & \omega^4 z_4 \end{matrix} \\ \boxed{\begin{matrix} \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \end{matrix}} \\ \begin{matrix} \omega^5 z_1 & \omega z_1 \end{matrix} \end{matrix}}_{D_4} \underbrace{\begin{matrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ \boxed{z_0} \end{matrix}}_{D_5} \right)$$

$$A_0 - 0 = \{0,1,2\}, A_4 - 4 = \{-1,0\}, A_5 - 5 = \{0\}.$$

Consider $\Lambda - (4,0)$.

$$\Lambda = \{(0,1),(0,5),(1,0),(2,0),(2,2),(2,3)\}.$$

$$A_0 = \{0,1\}, A_1 = \{2\}, A_2 = \{3,4,5\}.$$

$$D_\Lambda = \left(\begin{array}{cccccc} z_0 & z_0 & z_5 & z_4 & z_4 & z_4 \\ \omega z_1 & \omega^5 z_1 & z_0 & z_5 & \omega^2 z_5 & \omega^3 z_5 \\ \omega^2 z_2 & \omega^4 z_2 & z_1 & z_0 & \omega^4 z_0 & z_0 \\ \omega^3 z_3 & \omega^3 z_3 & z_2 & z_1 & z_1 & \omega^3 z_1 \\ \omega^4 z_4 & \omega^2 z_4 & z_3 & z_2 & \omega^2 z_2 & z_2 \\ \omega^5 z_5 & \omega z_5 & z_4 & z_3 & \omega^4 z_3 & \omega^3 z_3 \\ \multicolumn{2}{c}{\underbrace{\qquad\qquad\qquad}_{D_0}} & \underbrace{\quad}_{D_1} & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{D_2}} \end{array}\right)$$

$$A_0 = \{0,1\}, A_1 - 1 = \{1\}, A_2 - 2 = \{1,2,3\}.$$

So, $Z = Z^\iota = z_0 z_1^3 z_2 z_3$.

Theorem

Assume that all $A_\kappa - \kappa$ consist of nonnegative numbers; then $E[X] \leq E[X_\sigma]$ and $E[X^2] \leq E[X_\sigma^2]$ for all $\sigma \in S_N$. Furthermore, $E[X^2] = E[X_\sigma^2]$ if and only if σ is trivial.

Define $b_n = \kappa$ when $n \in A_\kappa$. By hypothesis, or $n - b_n \geq 0$, for all n. Then, the indices appearing in Z^σ are $\sigma(n) - b_n + \varepsilon N$ for $\varepsilon = 0$ or 1. So,

$$E[X_\sigma] \geq \frac{1}{N} \sum_{n=0}^{N-1} \sigma(n) - b_n = E[X].$$

Next, define

$$\sigma'(n) = \begin{cases} \sigma(n), & \text{if } \sigma(n) - b_n \geq 0 \\ \sigma(n) + N, & \text{if } \sigma(n) - b_n < 0, \end{cases}$$

so that

$$E[X_\sigma^2] = \frac{1}{N} \sum_{n=0}^{N-1} (\sigma'(n) - b_n)^2.$$

Let $f(n)$ be the unique strictly increasing *rearrangement* of σ'. Then

$$E[X_\sigma^2] - \frac{1}{N} \sum_{n=0}^{N-1} (f(n) - b_n)^2 = \frac{2}{N} \sum_{n=0}^{N-1} f(n) b_n - \frac{2}{N} \sum_{n=0}^{N-1} \sigma'(n) b_n \geq 0,$$

Also, $f(n) - b_n \geq n - b_n \geq 0$, so

$$\frac{1}{N}\sum_{n=0}^{N-1}(f(n) - b_n)^2 - E[X^2] \geq 0,$$

so, $E[X^2] \leq E[X_\sigma^2]$. If $E[X^2] = E[X_\sigma^2]$, then $f(n) = n$ for all n, so $\sigma(n) - b_n \geq 0$ for all n, hence

$$\begin{aligned} E[X_\sigma^2] - E[X^2] &= \frac{2}{N}\sum_{n=0}^{N-1} nb_n - \frac{2}{N}\sum_{n=0}^{N-1}\sigma(n)b_n \\ &= \frac{2}{N}\sum_{n=0}^{N-1} nb_n - \frac{2}{N}\sum_{n=0}^{N-1} nb_{\sigma^{-1}(n)} \end{aligned}$$

which is 0 if and only if σ is trivial.

Define

$$Q_\Lambda(x) = P_\Lambda(1, x, x^4, \dots, x^{(N-1)^2}) \in \mathbb{Q}(\omega)[x]$$

so

$$Z^\sigma = z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_{N-1}^{\alpha_{N-1}}$$

becomes

$$x^{\alpha_1} x^{2^2 \alpha_2} \cdots x^{(N-1)^2 \alpha_{N-1}} = x^{N \cdot E[X_\sigma^2]}$$

and

$$Q_\Lambda(x) = \sum_{\sigma \in S_N/\Gamma} c_\sigma x^{N \cdot E[X_\sigma^2]}$$

So $Q_\Lambda(x) \neq 0$ for all Λ and $\deg Q_\Lambda \leq N(N-1)^2$

Theorem

Let ξ be a transcendental number or an algebraic number whose degree over $\mathbb{Q}(\omega)$ is $> N(N-1)^2$. Then

$$(1, \xi, \xi^4, \ldots, \xi^{(N-1)^2})$$

generates a Gabor frame in general position.

Corollary

Let $N \geq 4$ and ζ be any primitive root of unity, of order $(N-1)^4$. Then

$$(1, \zeta, \zeta^4, \ldots, \zeta^{(N-1)^2})$$

generates a Gabor frame in general position.

What happens for Gabor frames over non-cyclic groups? For $\mathbb{Z}_2 \times \mathbb{Z}_2$ there are no Gabor frames in general position.