Moment problem for continuous linear functionals

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THE K-MOMENT PROBLEM

Let $A := \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ be the algebra of polynomials in n variables with real coefficients and $L : A \longrightarrow \mathbb{R}$ a real valued linear functional.

The K-moment problem

Given $\emptyset \neq K \subseteq \mathbb{R}^n$, when is L representable as an integral with respect to a positive Borel measure, i.e.

$$L(f) = \int_K f \, d\mu, \quad \forall f \in \mathbb{R}[\underline{X}],$$

where μ is supported on K?

THE K-MOMENT PROBLEM

Haviland, 1936

Such a measure exists if and only if $L(\mathrm{Psd}(K)) \subseteq [0, \infty)$, where $\mathrm{Psd}(K) := \{f \in A : f(x) \geq 0 \quad \forall x \in K\}$.

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Scheiderer, 1999

Except for a few cases, checking $L(\mathrm{Psd}(K)) \subseteq [0, \infty)$ is not a finite procedure, i.e. $\mathrm{Psd}(K)$ usually is not *finitely generated*.

DEFINITIONS

- $M \subseteq A$ is a quadratic module:
  - M is a cone:
$$0, 1 \in M, \quad M + M \subseteq M \text{ and } [0, \infty) \cdot M \subseteq M.$$
  - $\forall f \in A \quad f^2 M \subseteq M.$

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- M (or T) is finitely generated, if $M = M_S$ (or $T = T_S$) for some finite S.
- $S \subset A$:
$$\mathcal{K}_S := \{x \in \mathbb{R}^n \ : \ f(x) \geq 0 \quad \forall f \in S\}.$$

Classical solutions

Schmüdgen, 1991

If S is finite and $\mathcal{K}_S$ is compact, then

$$L(T_S) \subseteq [0, \infty) \Rightarrow L(\mathrm{Psd}(\mathcal{K}_S)) \subseteq [0, \infty).$$

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Since T_S and M_S are finitely generated, Haviland's Theorem is effectively applicable to them.

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- For a locally convex topology τ on A,
- C is a convex cone of A,
- and K is a closed subset of $\mathbb{R}^n$.

Example

1. Replace φ by $\|\cdot\|_K$-topology, where $K = [-1,1]^n$ and

$$\|f\|_K := \sup_{x \in K} |f(x)|.$$

Stone-Weierestrass $\Rightarrow$ $\mathrm{Psd}(K) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_K}$.

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2. Replace φ by $\|\cdot\|_1$-topology, $K=[-1,1]^n$ where
$$\|\sum_\alpha f_\alpha \underline{X}^\alpha\|_1 := \sum_\alpha |f_\alpha|.$$
Berg *et al.* $\Rightarrow$ $\mathrm{Psd}(K)=\overline{\sum\mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1}$.

Example

In term of moments:

If L is a $\|\cdot\|_K$ or $\|\cdot\|_1$- continuous positive semidefinite functional, then there exists a Borel measure μ on $[-1,1]^n$ such that

$$\forall f \in \mathbb{R}[\underline{X}] \quad L(f) = \int_{[-1,1]^n} f \, d\mu.$$

General settings:

Now, let A be a unital commutative $\mathbb{R}$-algebra and $\mathcal{X}(A) := \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R}) \subseteq \mathbb{R}^A$.

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Jacobi's Theorem, 2001

Let C be an Archimedean $\sum A^{2d}$-module of A. Then for each $a \in A$

$$\hat{a} > 0 \text{ on } \mathcal{K}_C \Rightarrow a \in C.$$

Seminormed Algebras

A map $\rho : A \longrightarrow [0, \infty)$ is called a seminorm if

1. $\forall a \in A \, \forall r \in \mathbb{R} \quad \rho(ra) = |r|\rho(a)$,
2. $\forall a, b \in A \quad \rho(a + b) \leq \rho(a) + \rho(b)$;

ρ is submultiplicative if

3. $\forall a, b \in A \quad \rho(ab) \leq \rho(a)\rho(b)$.

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is a compact Hausdorff space.

Seminormed Algebras

$I_\rho := \{a \in A \ : \ \rho(a) = 0\}$ is an *ideal* of A and

$$\begin{array}{rcl} \bar{\rho} : \bar{A} = A/I_\rho & \longrightarrow & [0, \infty) \\ \bar{a} & \mapsto & \rho(a) \end{array}$$

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Let $(B, \|\cdot\|)$ be a Banach algebra, $a \in B$, $r > \|a\|$ and $k \geq 1$ an integer. Then there exist $b \in B$ such that $b^k = r + a$.

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Proof.

The Tylor expansion of $(1+t)^{\frac{1}{k}} = \sum_{i=0}^{\infty} \lambda_i t^i$ converges absolutely for $|t| < 1$. Replace t with $\frac{a}{r}$, the conclusion follows. □

Main result

Theorem 1

Let (A, ρ) be a seminormed $\mathbb{R}$-algebra, $C \subseteq A$ a $\sum A^{2d}$-module and $d \geq 1$ an integer. Then

$$\overline{C}^{\rho} = \operatorname{Psd}(\mathcal{K}_C \cap \mathfrak{sp}(\rho)).$$

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Proof.
($\subseteq$): $C \subseteq \mathrm{Psd}(\mathcal{K}_C \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in \mathcal{K}_C \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0,\infty))$ which is closed.
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$$\forall n \geq 1 \; \forall \alpha \in \mathcal{K}_{\tilde{C}} \quad \alpha(\frac{1}{n} + \tilde{b}) > 0.$$

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By Jacobi's Theorem, $\tilde{b} + \frac{1}{n} \in \tilde{C}$. Letting $n \to \infty$, $\tilde{b} \in \tilde{C}$, and hence $b \in \overline{C}^{\rho}$. □

Corresponding moment problem

Corollary

Let $L : A \longrightarrow \mathbb{R}$ be a ρ-continuous linear functional. If $L(C) \subseteq [0, \infty)$ then there exists a Borel measure μ on $\mathcal{K}_C \cap \mathfrak{sp}(\rho)$ such that

$$L(a) = \int \hat{a}\, d\mu, \quad \forall a \in A.$$

Locally Multiplicatively Convex Topologies

Let $\mathcal{F}$ be a family of submultiplicative seminorms on A. The family $\mathcal{F}$ induces a locally convex topology $\tau_{\mathcal{F}}$ on A such that $(A, \tau_{\mathcal{F}})$ is a topological algebra.

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Proposition

If $\mathcal{F}$ is saturated then $\mathfrak{sp}(\tau_{\mathcal{F}}) = \bigcup_{\rho\in\mathcal{F}} \mathfrak{sp}(\rho)$.

Closures and moments in lmc topologies

Theorem 2

Let τ be an lmc topology on A, C a $\sum A^{2d}$-module and $d \geq 1$ an integer. Then

$$\overline{C}^{\tau} = \mathrm{Psd}(\mathcal{K}_C \cap \mathfrak{sp}(\tau)).$$

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Corollary

Let $L : A \longrightarrow \mathbb{R}$ be a τ-continuous functional with $L(C) \subseteq [0, \infty)$. Then there exists a Borel measure μ on $\mathcal{K}_C \cap \mathfrak{sp}(\tau)$ such that

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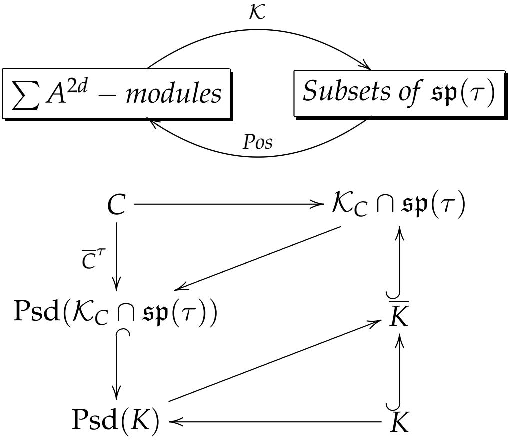

Schmüdgen's result

Schmüdgen, 1978

Let η be the finest lmc topology on A and $d \geq 1$. Then

$$\overline{\sum A^{2d}}^{\eta} = \mathrm{Psd}(\mathcal{X}(A)).$$

Involutive $\mathbb{C}$-algebras

Let $(A, \rho, *)$ be a seminormed $\mathbb{C}$-algebra equipped with an involution $*$.

INVOLUTIVE $\mathbb{C}$-ALGEBRAS

Let $(A, \rho, *)$ be a seminormed $\mathbb{C}$-algebra equipped with an involution $*$.

- $\mathcal{X}_*(A) := \{\alpha : A \longrightarrow \mathbb{C} \ : \ \alpha$ is a *-algebra homomorphism$\}$,
- $\mathfrak{sp}_*(\rho) := \{\alpha \in \mathcal{X}_*(A) \ : \ \alpha$ is ρ-continuous$\}$,
- $H(A) := \{a \in A \ : \ a^* = a\}$.

INVOLUTIVE $\mathbb{C}$-ALGEBRAS

Let $(A, \rho, *)$ be a seminormed $\mathbb{C}$-algebra equipped with an involution $*$.

- $\mathcal{X}_*(A) := \{\alpha : A \longrightarrow \mathbb{C} \ : \ \alpha$ is a *-algebra homomorphism$\}$,
- $\mathfrak{sp}_*(\rho) := \{\alpha \in \mathcal{X}_*(A) \ : \ \alpha$ is ρ-continuous$\}$,
- $H(A) := \{a \in A \ : \ a^* = a\}$.

Corollary

Let $C \subseteq H(A)$ be a $\sum H(A)^{2d}$-module of $H(A)$. Let $L : A \longrightarrow \mathbb{C}$ be a ρ-continuous $*$-functional such that $L(C) \subseteq [0, \infty)$. Then there exists a Borel measure μ on $\mathcal{K}_C \cap \mathfrak{sp}_*(\rho)$ such that

$$\forall a \in A \quad L(a) = \int \hat{a} \, d\mu.$$

Berg-Maserick

Let $(S, 1, *)$ be a commutative unitary $*$-semigroup.
An *absolute value* on S is a map $\phi : S \longrightarrow [0, \infty)$ such that

1. $\phi(1) \geq 1$,
2. $\forall s, t \in S$, $\phi(st) \leq \phi(s)\phi(t)$,
3. $\forall s \in S \quad \phi(s^*) = \phi(s)$.

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The map $\|\cdot\|_\phi$ on $\mathbb{C}[S]$ defined by $\|\sum_s f_s s\|_\phi = \sum_s |f_s|\phi(s)$ is a submultiplicative seminorm on $\mathbb{C}[S]$.

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Berg-Maserick, 1984

If $L : \mathbb{C}[S] \longrightarrow \mathbb{C}$ is an $*$-functional such that $L(\sum H(\mathbb{C}[S])^{2d}) \subseteq [0, \infty)$ and $\exists c > 0 \forall s \in S \ |L(s)| \leq c\phi(s)$. Then there exists a Borel measure μ on $\mathfrak{sp}_*(\|\cdot\|_\phi)$ $L(f) = \int \hat{f}\, d\mu$.