Ehrhart polynome: how to compute the highest degree coefficients and the knapsack problem.

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The two ingredients of the title

- Ehrhart (quasi)-polynome: the highest degree coefficients
- The case we want to study: the knapsack

Ehrhart polynome: the classical case

A rational polytope $\mathfrak{p} \subset \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ with rational coordinates.

Ehrhart quasi polynome computes the number of integral points in a polytope dilated by an integer in all the directions

- $\mathfrak{p}$ a convex polytope in $\mathbb{R}^d$
- $E(n, \mathfrak{p}) = |n\mathfrak{p} \cap \mathbb{Z}^d|$

The standard symplex Δ with vertices $0, e_1, e_2, ..., e_d$

The polytope $s\Delta$, $s \in \mathbb{N}$ defined by $x_1 \geq 0, ..., x_d \geq 0, x_1 + x_2 + \cdots + x_d \leq s$

- $|(s\Delta \cap \mathbb{Z}^d)| = \frac{(s+1)(s+2)...(s+d)}{d!}$
- $\text{vol}(s\Delta) = \frac{s^d}{d!}$

The knapsack problem

- $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{N+1}]$ a sequence of positive integers
- The simplex $\Delta_\alpha \subset \mathbb{R}^{N+1}$ defined by:
  $\Delta_\alpha = \{ [x_1, x_2, \ldots, x_N, x_{N+1}] : x_i \geq 0, \sum_{i=1}^{N+1} \alpha_i x_i = t \}$
- rational vertices $s_i = [0, \ldots, 0, \frac{t}{\alpha_i}, 0 \ldots, 0]$.
- knapsack problem:t a nonnegative integer, $E(\alpha)(t)$ the number of nonnegative solutions x_i of the equation $\sum_i \alpha_i x_i = t$
- $E(\alpha, t)$ is the Ehrhart (called *denumerant* in number theory) quasi polynome for Δ_α of degree N.
- quasi polynome: can be written in the form $E(\alpha)(t) = \sum_{i=0}^{N} E_i(t) t^i$, where $E_i(t)$ is a periodic function of t.
- $E(\alpha)(t) = 0$ if $t \notin \sum_{i=1}^{N+1} \mathbb{Z}\alpha_i \subset \mathbb{Z}$, lattice generated by the integers α_i.

Results

- A new algorithm that, for every fixed number k, computes in polynomial time the highest $k+1$ coefficients of the quasi-polynomial $E(\alpha)(t)$ as step polynomials of t.
- The determination of the highest degree for which this coefficient is truly periodic (not constant).

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May assume that the greatest common divisor of the α_i is equal to 1.
That is may assume numbers α_i span $\mathbb{Z}$ without changing the complexity of the problem.
Indeed if g is the greatest common divisor of the α_i (which can be computed in polynomial time) then $E(\alpha, gt) = E(\alpha/g, t)$

Example: the triangle

$\alpha = [6, 2, 3]$, the number of integer solutions to

$$6x + 2y + 3z = 6$$

is shown here:

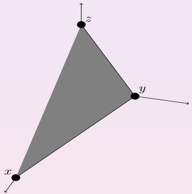

Figure: Polytope and integers points

The linear equation intersected with the first quadrant represents a polytope and the integer solutions of the equation are the integers points in the polytope.

Number of integer solutions of the knapsack problem $6x + 2y + 3z = 6$ is 3

The general formula: not so easy....

How to present the answer?

Our example continued

- $6x + 2y + 3z = t$
- It is a polytope of dimension 2 and vertices $[t/6, 0, 0], [0, t/2, 0], [0, 0, t/3]$

First way to present the answer

- $E(\alpha)(t)$ is a polynome in t whose coefficients are periodic (a function of t mod Q, where Q is a period to be computed).
- In our example $Q = 6$
  Then on each of the cosets $q + 6\mathbb{Z}$, the quasipolynomial function $E(\alpha)(t)$ coincides with a polynomial $E^{[q]}(t)$, which are given by these corresponding polynomials.

We compute the knapsack as function on cosets: $t \mod 6$

$$E^{[0]}(t) = \frac{1}{72}t^2 + \frac{1}{4}t + 1, \qquad E^{[1]}(t) = \frac{1}{72}t^2 + \frac{1}{18}t - \frac{5}{72},$$
$$E^{[2]}(t) = \frac{1}{72}t^2 + \frac{7}{36}t + \frac{5}{9}, \qquad E^{[3]}(t) = \frac{1}{72}t^2 + \frac{1}{6}t + \frac{3}{8},$$
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$E(\alpha)(6) = E^{[6 \mod 6]}(6) = E^{[0]}(6) = \frac{1}{2} + \frac{3}{2} + 1 = 3.$

We can also compute

$E(\alpha)(7) = E^{[1]}(7) = \frac{49}{72} + \frac{7}{18} - \frac{5}{72} = 1$ and

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Some comments

The number Q of cosets for $E(\alpha)(t)$ can be exponential in the binary encoding size of the problem, and thus it is impossible to list in polynomial time the polynomials $E^{[q]}(t)$ for all the cosets $q + Q\mathbb{Z}$. We need to present the same information in a more economical way: *step polynomials*.

Step polynomials

- Let $\{s\} := s - \lfloor s \rfloor \in [0, 1)$ for $s \in \mathbb{R}$.
- $\{s\}$ is the *fractional part* of s, it is periodic of period 1: $\{s\} = \{s + 1\}$
- If $r \in \mathbb{Q}$ with denominator q, the function $T \mapsto \{rT\}$ is periodic in $T \in \mathbb{R}$ modulo q.
- A *step polynomial* $\phi(T)$ will have the (non unique) form

$$\phi(T) = \sum_{l=1}^{L} c_l \prod_{j=1}^{J_l} \{r_{l,j} T\}^{n_{l,j}}.$$

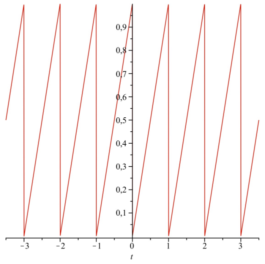

Figure: The fractional part $\{s\}$

Second way to present the answer

We compute for $\alpha = [6, 2, 3]$:

$E(\alpha)(t) = (1 - \frac{3}{2}\{\frac{2}{3}t\} - \frac{3}{2}\{\frac{1}{2}t\} + \frac{1}{2}\{\frac{2}{3}t\}^2 + \{\frac{2}{3}t\}\{\frac{1}{2}t\} + \frac{1}{2}\{\frac{1}{2}t\}^2) + (\frac{1}{4} - \frac{1}{6}\{\frac{2}{3}t\} - \frac{1}{6}\{\frac{1}{2}t\})t + \frac{1}{72}t^2$

In particular

$E(\alpha)(6) = 1 + \frac{1}{4}6 + \frac{1}{72}6^2 = 1 + 2 = 3$ and

$E(\alpha)(7) = (1 - \frac{3}{2}\{\frac{2\cdot 7}{3}\} - \frac{3}{2}\{\frac{7}{2}\} + \frac{1}{2}\{\frac{2\cdot 7}{3}\}^2 + \{\frac{2\cdot 7}{3}\}\{\frac{7}{2}\} + \frac{1}{2}\{\frac{7}{2}\}^2) + (\frac{1}{4} - \frac{1}{6}\{\frac{2\cdot 7}{3}\} - \frac{1}{6}\{\frac{7}{2}\})7 + \frac{7^2}{72} = (1 - \frac{3}{2}\cdot\frac{2}{3} - \frac{3}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot(\frac{2}{3})^2 + \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot(\frac{1}{2})^2) + (\frac{1}{4} - \frac{1}{6}\cdot\frac{2}{3} - \frac{1}{6}\cdot\frac{1}{2})7 + \frac{49}{72} = -\frac{5}{72} + \frac{7}{18} + \frac{49}{72} = 1$ and

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It is clear that the step functions allows a ”compact” writing of the results (so the polynomiality) on the other hand the first answer is easier to read but ”not short” .

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A closer look

Convex polytope: in $\mathbb{R}^N$

- A an r by N matrix with column vectors $\Phi = [\alpha_1, \alpha_2, \dots, \alpha_N]$
- $\mathcal{P}_\Phi(h) = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N \,|\, Ax = h,\ x_i \geq 0\right\},\ h \in \mathbb{R}^r$ is the convex polytope associate to Φ and h.

$\mathcal{P}_\Phi(h)$ consists of all the nonnegative solutions of the system of the r linear equations $\sum_{i=1}^N x_i\alpha_i = h$.

$\mathcal{P}_\Phi(h)$ is empty unless h is in the convex cone $\mathcal{C}(\Phi) = \{\sum \lambda_i\alpha_i, \lambda_i \in \mathbb{R}^+\} \subset \mathbb{R}^r$

- Polytope integer: the vertices have integer coordinates.
- Polytope rational: the vertices have rational coordinates.

First ingredient: Ehrhart quasipolynomial

- $\mathfrak{p}$ rational polytope in $\mathbb{R}^d$.
- $h(x)$ polynomial function on $\mathbb{R}^d$ with rational coefficients.
- $n \in \mathbb{N}$ dilatation factor

Ehrhart quasipolynomial: weighted case

$$E(n, \mathfrak{p}) := \sum_{x \in n\mathfrak{p} \cap \mathbb{Z}^d} h(x)$$

unweigthed case - If $h = 1$ we compute the number of integral points in the polytope: $E(n, \mathfrak{p}) = |n\mathfrak{p} \cap \mathbb{Z}^d|$

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Theorem (Ehrhart 1962).

- $n \mapsto E(n, \mathfrak{p})$ is a quasipolynomial,
- $E(n, \mathfrak{p}) = \sum_{k=0}^{\dim \mathfrak{p} + \deg h} E_k(n) n^k$,
- $E_k(n)$ depend only on $n \mod Q$, Q such that $Q\mathfrak{p}$ has vertices in $\mathbb{Z}^d$,
- $E_{\dim \mathfrak{p} + \deg h} = \int_{\mathfrak{p}} h(x)$ (if $h(x)$ homogeneous).

Ehrhart: one example

$$\sum_{x \in [0, \frac{n}{2}] \cap \mathbb{Z}} x = \begin{cases} \frac{n^2}{8} + \frac{n}{4} & \text{for } n \text{ even} \\ \frac{n^2}{8} - \frac{1}{8} & \text{for } n \text{ odd} \end{cases}$$

In particular if the polynome $x = 1$ we have

$$|[0, \frac{n}{2}] \cap \mathbb{Z}| = \begin{cases} \frac{n}{2} + 1 & \text{for } n \text{ even} \\ \frac{n}{2} + \frac{1}{2} & \text{for } n \text{ odd} \end{cases}$$

What can we *really* compute?

We are interested not only in exact computations but in the problem of ”counting” efficiently..
In other words we want that whatever we can compute can be implemented with algorithms of polynomial complexity.
As we already remarked the polynomiality depends also by the way we write the output.
Again for a given coset $q + Q\mathbb{Z}$, computing the polynomial $E^{[q]}(t)$ is NP-hard. So we cannot certainly write an answer in general as we did for our example in polynomial time. We do have to restrict our ambitions.

Polynomial time algorithms: status of the art.

- Computing number of lattice points of a simplex is *NP*-hard if the dimension is not fixed.
- Polynomial time algorithm for computing number of points of polytope when the dimension is fixed (1994- Barvinok) (implemented by LaTTe (De Loera et al 2004).
- Fixing k (but not the dimension) polynomial time algorithm for computing the top $k+1$ coefficients for a number of lattice points in a simplex. (Barvinok-2006)
- Fixing k and degree of h (but not the dimension) polynomial time algorithm for computing the top $k+1$ coefficients of the Ehrhart weighted quasi polynomial of a simplex (B.,Berline, De Loera, Koeppe, Vergne -2010)
- If N is fixed , Lenstra, decide in polynomial time if the knapsack problem has a solution for a given t.

Polynomial time algorithm for the following problem: B.,Berline, De Loera, Koeppe, Vergne

Fix M and $k_0 \in N$.

Input:

$\mathfrak{p}$ rational simplex in $\mathbb{R}^d$,

$h \in \mathbb{Q}[x_1, \ldots, x_d]$ with deg $h \leq M$.

Output:

The $k_0 + 1$ top degree Ehrhart coefficients $E_{\dim \mathfrak{p} + \deg h - k}$, $0 \leq k \leq k_0$ of

$$\sum_{x \in n\mathfrak{p} \cap \mathbb{Z}^d} h(x) = \sum_{k=0}^{\dim \mathfrak{p} + \deg h} E_k(n) n^k$$

in particular $\int_{\mathfrak{p}} h(x)$.

dim($\mathfrak{p}$) not fixed.

Conclusions in general and in particular for the knapsack problem

- To compute the full quasi-polynomial $E(\alpha)(t)$ is NP-hard. Thus we restrict our goal and we consider a reasonable problem from the algorithmic point of view.
- Given an integer k, our goal is to give an algorithm of polynomial complexity to compute the highest $k+1$ degree terms of the quasi polynomial $E(\alpha)(t)$, that is $Topk(E(\alpha)(t) = \sum_{i=0}^{k} E_{N-i}(t)t^{N-i}$
- We can consider this is an approximation of $E(\alpha)(t)$ since at least the highest degree coefficient is the approximation given by the volume. The coefficients are recovered as step polynomial functions of t.
- The top coefficient of $E(\alpha)(t)$ is the volume-easy: $\frac{t^N}{N! \prod_i \alpha_i}$ then the coefficients are more and more difficult to compute.....

A very important issue: how can the output be given?

Taking care of the polynomial complexity we thus concentrate on the problem of computing only the top $k+1$ coefficients.

- For a given coset $q + Q\mathbb{Z}$, computing the polynomial $E^{[q]}(t)$ is NP-hard.
- For any fixed number k, Barvinok gives a polynomial-time algorithm that, given an N-dimensional simplex Δ with rational vertices and a coset $q + Q\mathbb{Z}$, computes the highest $k+1$ coefficients of the polynomial $E^{[q]}(t)$ that coincides with the Ehrhart quasi-polynomial $E(\Delta)(t)$ of Δ for t in the coset $q + Q\mathbb{Z}$.
- Note that the number Q of cosets is already exponential in the binary encoding size of the problem, and thus it is impossible to list, in polynomial time, the polynomials $E^{[q]}(t)$ for all the cosets $q + Q\mathbb{Z}$.

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Conclusion: if one is interested in obtaining polynomial time algorithms then the output must have "compact form". We have an algorithm of polynomial complexity to compute the coefficient functions of $\mathrm{Top}_k E(P)(t)$ for any simple polytope P (given by its rational vertices) in the form of *step polynomials* .

The knapsack problem: a new approach for computing the top $k+1$ terms in polynomial time

The new approach is very simple to describe and it has a Maple implementation that is really very promizing respect to other methods.

- it is based on the one dimensional residue theorem
- Barvinok signed unimodular cone decomposition for simplicial cones in a vector space of fixed dimension (2005)
- The method is of polynomial complexity. It rests of course on the polynomial complexity of the cone dec in unimodular

knapsack: the coefficients-The example $6x + 2y + 3z = t$ continued

- Write $Top_k(E(\alpha)(t) = \sum_{i=0}^{k} E_{N-i}(t)t^{N-i}$
- $E_{N-i}(t)$ is periodic mod 6 given by
- $E(N,0) =$ $1 - \frac{3}{2}\left\{\frac{2}{3}t\right\} - \frac{3}{2}\left\{\frac{1}{2}t\right\} + \frac{1}{2}\left\{\frac{2}{3}t\right\}^2 + \left\{\frac{2}{3}t\right\}\left\{\frac{1}{2}t\right\} + \frac{1}{2}\left\{\frac{1}{2}t\right\}^2$
- $E(N,1) = \frac{1}{4} - \frac{1}{6}\left\{\frac{2}{3}t\right\} - \frac{1}{6}\left\{\frac{1}{2}t\right\}$
- $E(N,2) = \frac{1}{72}$

- Barvinok decomposition is polynomial because it is signed..
- we can partly reconduce our computation to a cone of dimension $\leq k$ (k is our fixed entry) and thus apply Barvinok decomposition to this cone.

A residue formula for $E(\alpha)(t)$: the function to study

Datas

- $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{N+1}]$ our list of integers with greates common divisor 1.
- $F(\alpha)(z) := \frac{1}{\prod_{i=1}^{N+1}(1-z^{\alpha_i})}$.
- $\mathcal{P} = \bigcup_{i=1}^{N+1}\{\, \zeta \in \mathbb{C} : \zeta^{\alpha_i} = 1 \,\}$ set of poles of the meromorphic function $F(\alpha)$ and by $p(\zeta)$ the order of the pole ζ for $\zeta \in \mathcal{P}$. 1 has order $N+1$ the other strictly less

- If $|z|$ small write $\frac{1}{1-z^{\alpha_i}} = \sum_{u=0}^{\infty} z^{u\alpha_i}$ to get $F(\alpha)(z) = \sum_{t\geq 0} E(\alpha)(t) z^t$.
- if t is a non-negative integer then $E(\alpha)(t) = \frac{1}{2i\pi}\int_{|z|=\epsilon} z^{-t-1}F(\alpha)(z)$
- apply the residue theorem to get $E(\alpha)(t) = -\sum_{\zeta\in\mathcal{P}} \mathrm{Res}_{z=\zeta} z^{-t-1}F(\alpha)(z)$
- the ζ-term of this sum is a quasi-polynomial function of t with degree less than or equal to $p(\zeta)-1$.

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- $\mathcal{P} = \bigcup_{i=1}^{N+1}\{\, \zeta \in \mathbb{C} : \zeta^{\alpha_i} = 1 \,\}$ set of poles of the meromorphic function $F(\alpha)$ and by $p(\zeta)$ the order of the pole ζ for $\zeta \in \mathcal{P}$. 1 has order $N+1$ the other strictly less

- If $|z|$ small write $\frac{1}{1-z^{\alpha_i}} = \sum_{u=0}^{\infty} z^{u\alpha_i}$ to get $F(\alpha)(z) = \sum_{t\geq 0} E(\alpha)(t)z^t$.
- if t is a non-negative integer then $E(\alpha)(t) = \frac{1}{2i\pi}\int_{|z|=\epsilon} z^{-t-1}F(\alpha)(z)$
- apply the residue theorem to get $E(\alpha)(t) = -\sum_{\zeta\in\mathcal{P}} \mathrm{Res}_{z=\zeta} z^{-t-1}F(\alpha)(z)$
- the ζ-term of this sum is a quasi-polynomial function of t with degree less than or equal to $p(\zeta) - 1$.

A residue formula for $E(\alpha)(t)$: the function to study

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Thus to compute the highest degree coefficients we need only poles of high order!!!!! So we want only the ζ's for which $\zeta^{\alpha_i} = 1$ for many α_i.
When we look only at the contribution of this poles then we compute in polynomial time as we will see.

How do we deal with the poles?

- Partition the poles in two disjoint set according to the order of poles...
- Use a Moebius function to reduce the situation to compute a sum of terms as $E(\alpha, f)(t) = -\sum_{\zeta^f=1} \mathrm{Res}_{z=\zeta} z^{-t-1} F(\alpha)(z)$, f is a positive integer and we need only the f for which the pole of ζ is $\geq N - k + 1$.

Such an f is the greatest common divisor of a sublist of α and $\zeta^f = 1$ says that ζ has a pole of order $\geq l$, where l is the lenght of the list.

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Example: knapsack problem for coefficients $[6, 2, 3]$, $N = 2$, $k = 1$.

- $\zeta_6 = e^{2i\pi/6}$ primitive 6-th root of unity. So $\zeta_6^6 = 1$ is a pole of order 3 , ζ_6 and ζ_6^5 are poles of order 1, and ζ_6^2 and ζ_6^4 are poles of order 3, while , $\zeta_6^3 = -1$,.
- Thus the poles of order greater then 1 are $\{\,\zeta : \zeta^2 = 1\,\} = \{-1, 1\}$ and $\{\,\zeta : \zeta^3 = 1\,\} = \{\zeta_6^2, \zeta_6^4, \zeta_6^6 = 1\}$.

$\mathcal{P} = \mathcal{G}(2) \cup \mathcal{G}(3)$, $\mathcal{G}(f) = \{\,\zeta \in \mathbb{C} : \zeta^f = 1\,\}$,
The decomposition via Moebius is ([ ] characteristic functions of the sets $\mathcal{G}(f)$):
$[\mathcal{P}] = -[\mathcal{G}(1)] + [\mathcal{G}(2)] + [\mathcal{G}(3)]$ since $\mathcal{G}(1)$ is both a subset of $\mathcal{G}(2)$ and $\mathcal{G}(3)$, which gives
$E(\alpha)(t) = -E(\alpha, 1)(t) + E(\alpha, 2)(t) + E(\alpha, 3)(t)$

we continue..

- $E(\alpha, f)(t) = -\sum_{\zeta^f=1} \mathrm{Res}_{z=\zeta} z^{-t-1} F(\alpha)(z)$t
- Writing $z = \zeta e^x$ and changing coordinates in residues, rewrite $E(\alpha, f)(t) =$
$-\mathrm{res}_{x=0} e^{-Tx} \sum_{\zeta^f=1} \frac{\zeta^{-T}}{\prod_{i=1}^{N+1}(1-\zeta^{\alpha_i} e^{\alpha_i x})}\Big|_{T=t} = \mathcal{E}(\alpha, f)(T)\Big|_{T=t}$

Recall that we want the degree coefficients $t^N, \cdots, t^{N-k}$, so we want f such that the poles of ζ satisfies $p(\zeta) \geq N - k + 1$, that is there are at least $N - k + 1$ elements α_i such that $\zeta^{\alpha_i} = 1$. Riarranging the indices we can rewrite

$$\mathcal{E}(\alpha, f)(T) =$$

$$-\mathrm{res}_{x=0} e^{-Tx} \sum_{\zeta^f=1} \frac{\zeta^{-T}}{\prod_{i=0}^{N-s}(1-\zeta^{\alpha_i} e^{\alpha_i x}) \prod_{i=N-s+1}^{N}(1-e^{\alpha_i x})}$$

where $s \geq N - k + 1$.

Theorem

Let k be fixed and f integer so that $p(\zeta) > N - k + 1$ then

$$\mathcal{E}(\alpha, f)(t, T) = \sum_{i=0}^{N} t^i E_i(f)(T)$$

with $E_i(f)(T)$ a step polynomial of degree less than or equal to $N - i$ and periodic of T modulo f. This step polynomial can be computed in polynomial time.

For $0 \leq k \leq N$, the coefficient of t^k in the Ehrhart quasi-polynomial is given by

$$E_k(\alpha)(t) = -res_{x=0} \frac{(-x)^k}{k!} \sum_{f \in \mathcal{G}_{>k}(\alpha)} \mu(f) \sum_{\{\zeta;\zeta^f=1\}} \frac{\zeta^{-t}}{\prod(1 - \zeta^{\alpha_i} e^{\alpha_i x})}$$

where $\mathcal{G}_{>k}$ parametrize the greatest common divisor of sublists of the α's of lenght $\geq k + 1$.

The last ingredient: Barvinok cone unimodular decomposition

Fix k and consider a cone in $\mathbb{R}^k$ and a lattice Λ.
Then the function $S(C,\Lambda)(x) = \sum_{n \in C \cap \Lambda} e^{\sum n_j x_j}, x_1, \cdots, x_k \in \mathbb{R}$ can be computed in polynomial time.
In the case of $\mathbb{R}$ with lattice $\mathbb{Z}$, $C = \mathbb{R}_{\geq 0}$ we simply have $S(\mathbb{R}_{\geq 0}, \mathbb{Z})(x) = \sum_{n \geq 0} e^{nx}$
Any cone can be decomposed into a signed sum of unimodular cones modulo cones which contains lines and the decomposition can be done in polynomial time (Barvinok 2005)
So we can compute the above sum over unimodular cone. So the formula is just product of the formula in $\mathbb{R}$.

One more step to go...

Recall our function: (we have simplified using m instead then $N-s+1$ with $m \le k$)

$$\mathcal{E}(\alpha, f)(T) = -\mathrm{res}_{x=0} e^{-Tx} \sum_{\zeta^f=1} \frac{\zeta^{-T}}{\prod_{i=0}^{m-1}(1-\zeta^{\alpha_i} e^{\alpha_i x}) \prod_{i=m}^{N}(1-e^{\alpha_i x})}$$

To simplify the exposition let's compute on the coset $T = 0 \mod f$ (we are taking the case in which the cone has vertex 0).

$$\mathcal{E}(\alpha, f)(T) = -\mathrm{res}_{x=0} e^{-Tx} \sum_{\zeta^f=1} \frac{1}{\prod_{i=0}^{m-1}(1-\zeta^{\alpha_i} e^{\alpha_i x}) \prod_{i=m}^{N}(1-e^{\alpha_i x})}$$

$\sum_{\zeta^f=1} \frac{1}{\prod_{i=1}^{k}(1-\zeta^{\alpha_i} e^{\alpha_i x})}$ is the generating function of the standard cone over a sublattice of $\mathbb{Z}^k$: $\Lambda(\alpha, f) = \{y \in \mathbb{Z}^k : \sum_j y_j \alpha_j \in \mathbb{Z}f\}$

Indeed

$$\sum_{\zeta^f=1} \frac{1}{\prod_{i=1}^{k}(1-\zeta^{\alpha_i} e^{\alpha_i x})}$$

$$= \sum_{[n_1 \cdots, n_k] \in \mathbb{Z}^k} \sum_{\zeta^f=1} \zeta^{\sum_i n_i \alpha_i} e^{\sum_i n_i \alpha_i x} = S(\mathbb{R}^k_{\ge 0}, \Lambda(\alpha, f))(x)$$

since $\sum_{\zeta^f=1} \zeta^p$ is zero except if $p \in \mathbb{Z}f$, when this sum is equal to f.

To conclude we only.. have to take at most k terms of a Fourier series expansion to compute the first $k+1$ coefficients...

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The result on the periodicity

We recall the datas

- $\alpha := [\alpha_1, \alpha_2, \ldots, \alpha_{N+1}]$ with greatest common divisor equal to 1,
- $F_\alpha(z) := \frac{1}{\prod_{i=1}^{N+1}(1-z^{\alpha_i})}$.
- $E(\alpha)(t) = -\sum_{\zeta\in\mathcal{P}} Res_{z=\zeta} z^{-t-1} F_\alpha(z)dz$ with t a non negative integer
- Then the ζ-term of this sum is a quasi polynomial function of t with degree less or equal to $p(\zeta) - 1$.
- Moreover for k such that $0 \leq k \leq N$, the coefficient of t^k in the Ehrhart quasi-polynomial is given by $E_k(\alpha)(t) = -res_{x=0} \frac{(-x)^k}{k!} \sum_{f\in\mathcal{G}_{>k}(\alpha)} \mu(f) \sum_{\{\zeta;\zeta^f=1\}} \frac{\zeta^{-t}}{\prod(1-\zeta^{\alpha_i} e^{\alpha_i x})}$

Determination of the highest degree for which the coefficient is truly periodic.

Theorem

Let ℓ be the greatest integer for which there exists a sublist α_J with $|J| = \ell$, such that its gcd f is not 1. Then for $k \geq \ell$ the coefficient of degree k, $E_k(\alpha)(t)$, is constant while the coefficient of degree $\ell - 1$, $E_{\ell-1}(\alpha)(t)$, is strictly periodic.

Experiments

We wrote a version of our algorithm in `Maple`, and tested it on a database of 140 knapsacks among four classes for the knapsack's hyperplane a:

- partition: $a = [1, 2, 3, \ldots, N + 1]$
- random 3 (or 15): each coefficient α_i is a 3- (or 15-) digit random number,
- random 15: each coefficient α_i is a 15-digit random number,
- repeat: $\alpha_1 = 1$ and all the other α_i's are the same 3-digit random number,

We limited the computation time for a coefficient to two minutes. Additional experiments with other software tools appear in a forthcoming paper.