Some Basis Theorems in Recursion Theory

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Cantor and Baire Space

Definition

1. Cantor space is a topology over 2^ω. The basic open sets are those sets of reals extending a fixed finite string $\sigma \in 2^{<\omega}$. We use $[\sigma]$ to denote the basic open set $\{x \in 2^\omega \mid x \succ \sigma\}$.
2. Baire space is a topology over ω^ω. The basic open sets are those sets of reals extending a fixed finite string $\sigma \in \omega^{<\omega}$. We use $[\sigma]$ to denote the basic open set $\{x \in \omega^\omega \mid x \succ \sigma\}$.

König Lemma

Lemma

If T is a finitely branching tree, then $[T] = \{x \mid x$ is an infinite path through $T\}$ is not empty if and only if T is infinite.

Proof.

"$\rightarrow$" is trivial.
"$\leftarrow$. Suppose T is infinite. Then using the fact as an "oracle" to construct an infinite path. □

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Exercise

- *Cantor space is compact.*
- *König lemma fails if T is not finitely branching.*

Kreisel's Basis Theorem

Theorem

If $T \subseteq 2^{<\omega}$ is an infinite recursive tree, then T has an infinite path recursive in $\emptyset'$.

Proof.

Let x be the leftmost path of T. The existence of x is guaranteed by König lemma.
Then $x \leq_T \emptyset'$. □

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Exercise

There is an infinite recursive tree $T \subseteq 2^{<\omega}$ so that T has no recursive infinite path.

Relativization and Lowness

A real $x \in 2^\omega$ is GL_1 if $x' \equiv_T x \oplus \emptyset'$.
A real x is low (L_1) if $x \leq_T \emptyset'$ and x is GL_1.

A Nonrecursive Low Real

Theorem

There is a nonrecursive low real.

Proof.

$$N_e : \Phi_e \text{ is total} \implies x \neq \Phi_e.$$

We satisfy N_e to ensure x is nonrecursive.
To make it be low, we need $\{e \mid \Phi_e^x(e)\downarrow\} \leq_T \emptyset'$.
We perform a $\emptyset'$-effective Cohen-forcing construction to make x be GL_1.

□

Jockusch-Soare Forcing

Definition

A JS forcing is a partial ordering $\langle \mathbf{T}, \leq \rangle$ where $\mathbf{T} = \{T \subseteq 2^{<\omega} \mid T \text{ is an infinite recursive tree}\}$. $T_1 \leq T_2$ if $T_1 \subseteq T_2$.

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Exercise

There is an r.e. infinite tree T so that every infinite path in T computes the halting problem.

Low Basis Theorem

Theorem

If T is a condition, then there is a real $x \in [T]$ so that $x' \leq_T \emptyset'$.

Proof.

An $\emptyset'$-effective forcing argument.
For any e and condition S, check whether there is another condition $S_1 \leq S$ so that for any $\sigma \in S_1$, $\Phi_e^\sigma(e) \uparrow$. □

Hyperimmune-freeness

Definition

A real x is hyperimmune-free if for any function $f \leq_T x$, there is a recursive function g dominating f. In other words, $\forall n(g(n) > f(n))$.

Theorem

If $x \leq_T \emptyset'$ is not recursive, then x is not hyperimmune-free.

Proof.

By Shoenfield lemma, there is a recursive function f so that $x(n) = \lim_{s\to\infty} f(n,s)$.
Let $g(n)$ be the least $s_n \geq n$ so that for any $m \leq n$, $x(m) = f(m, s_n)$. Then $g(n)$ is not dominated by any recursive function. □

Exercise

1. *If x is r.e. in $\emptyset'$ and hyperimmune-free, then x is recursive.*
2. *There is a nonrecursive hyperimmune-free real $x \leq_T \emptyset''$.*

Hyperimmune-free Basis Theorem

Theorem

If T is a condition, then there is hyperimmune-free real $x \in [T]$.

Proof.

A forcing construction.
Check whether there is some n so that there is some $S \leq T$ and $S \Vdash \Phi_e(n) \uparrow$. □

Peano Arithmetic Theory

Definition

x has Peano degree, or PA, if there is a theory $T \leq_T x$ which is a complete extension of Peano arithmetic axioms.

By Gödel incompleteness, no PA degree can be recursive.

A set of *PA*-degrees

Theorem

There is a condition T so that any $x \in [T]$ is of PA.

So there is a *PA*-real x so that $x' \leq_T \emptyset'$.

PA-basis Theorem

Theorem

If T is a condition and x is of PA, then there is some $y \in [T]$ so that $y \leq_T x$.

Proof.

First, given any number n, x proves that T has height at least n. By induction, given $\sigma_n \in T$, decide the sentence "$\sigma_n^\frown 0$ has height less than $\sigma_n^\frown 1$ in T". □

Exercise

1. *Prove that if x is of PA, then there is some real $y \leq_T x$ so that $\forall e(\Phi_e(e) \not\simeq y(e))$.*
2. *If x is of PA, then there is a complete extension T of PA so that $T \equiv_T x$.*

On Π^1_1-sets

There is an effective enumeration of recursive trees in $\omega^{<\omega}$. In other words, there is a recursive set $U \subseteq \omega \times \omega^{<\omega}$ so that for any e, $U_e = \{\sigma \mid (e, \sigma) \in U\}$ is a tree and for every recursive tree T, there is some U_e so that $[U_e] = [T]$.
Let $WF = \{e \mid U_e \text{ has no infinite path}\}$.
Obviously WF is a Π^1_1-set.

Π^1_1-completeness

Theorem

WF is Π^1_1-complete.

Proof.

We need to prove that given any Π^1_1-real x, there is a recursive function f so that $x(n) = 1 \leftrightarrow n \in WF$. □

Corollary

WF is not Σ^1_1.

Proof.

Otherwise, then $\forall n(n \notin WF \leftrightarrow f(n) \in WF)$ for some recursive function f. A contradiction by recursion theorem. □

Gandy Basis Theroem

Theorem

1. *If $T \subseteq \omega^{<\omega}$ has an infinite path, then there is some $x \in [T]$ so that $x \leq_T WF$.*
2. *If $A \subseteq 2^\omega$ is a nonempty Σ^1_1 set, then A contains a real recursive in WF.*

Proof.

By induction recursive in WF. □

Exercise

Prove that there is recursive tree $T \subseteq \omega^{<\omega}$ having an infinite path but no path recursive in $\emptyset'$.

Kleene-Brouwer Ordering

Given $\sigma, \tau \in \omega^{<\omega}$, we say that $\sigma <_{KB} \tau$ if either $\sigma \succ \tau$ or $\exists n(\sigma \upharpoonright n = \tau \upharpoonright n \wedge \sigma(n) < \tau(n))$.

Lemma

A tree T has no infinite path if and only if $<_{KB}$ is a T-recursive well ordering over T.

Proof.

"$\leftarrow$" is obvious.
"$\rightarrow$" can be proved as in König lemma. □

Representations of Π^1_1-sets

Theorem

A set of reals $A \subseteq 2^{<\omega}$ *is* Π^1_1 *if and only if there is a recursive tree* $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ *so that*
$x \in A \leftrightarrow T_x = \{\sigma \mid \exists n((x \upharpoonright n, \sigma) \in T)\}$ *has no infinite path.*

So $x \in A$ if and only if $<_{KB}$ is a well ordering over x-recursive tree T_x.

Constructibility (1)

$$L_0 = \emptyset;$$

$$L_{\alpha+1} = \{x \mid \exists\varphi(z \in x \leftrightarrow L_\alpha \models \varphi(z))\};$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\lambda;$$

$$L = \bigcup_{\alpha} L_\alpha.$$

Gödel proves that L is a model of ZFC.

Constructibility (2)

Given a set y,

$$L_0[y] = \emptyset;$$

$$L_{\alpha+1}[y] = \{x \mid \exists\varphi(z \in x \leftrightarrow \langle L_\alpha[y], y \cap L_\alpha[y]\rangle \models \varphi(z, y \cap L_\alpha[y]))\};$$

$$L_\lambda[y] = \bigcup_{\alpha<\lambda} L_\lambda[y];$$

$$L[y] = \bigcup_\alpha L_\alpha[y].$$

Exercise

Prove that if $x \in 2^\omega$ does not belong to L, then $x \in L[x]$ but $L[\{x\}] = L$.

Representing Π^1_1 sets in L

Theorem

If $A \subseteq 2^{<\omega}$ is Π^1_1, then there is tree $T \subseteq 2^{<\omega} \times \omega_1^{<\omega}$ in L so that $x \in A \leftrightarrow \exists f \in \omega_1^\omega \cap L[x]((x, f) \in [T])$.

Proof.

By the presentation of Π^1_1-sets, let S be the recursive tree.
Let $(\sigma, \tau) \in T$ if τ is a finite order preserving (in the $<_{KB}$ sense) function from $|\sigma|^{|\sigma|} \cap \{\nu \mid (\sigma, \nu) \in S\}$ to ω_1 $\square$

A Basis Theorem for Π^1_1-set

Theorem

If $A \subseteq 2^\omega$ is Π^1_1, then there is some real $x \in A$ so that $x \in L$.

Proof.

Presenting Π^1_1-sets as a tree T in L. Consider the left most infinite path in T. □

Exercise

Prove that there is a Π^1_1 set $\{x\} \subset 2^\omega$ so that x is not recursive in WF.

Shoenfield Absoluteness

Theorem

If $A \subseteq 2^\omega$ is Σ^1_2, then there is some real $x \in A$ so that $x \in L$.

Proof.

Immediately from the previous theorem. □

Exercise

1. *$A \subseteq 2^{<\omega}$ is Σ^1_2 if and only if there is tree $T \subseteq 2^{<\omega} \times \omega_1^{<\omega}$ in L so that $x \in A \leftrightarrow \exists f \in \omega_1^\omega \cap L[x]((x, f) \in [T])$.*
2. *$x \in 2^\omega$ is Σ^1_2 if and only if there is Σ_1-formula φ so that $n \in x \leftrightarrow L_{\omega_1^L} \models \varphi(n)$.*

Thanks