Steady vortex solutions for Euler equation of two dimension

Daomin Cao
Institute of Applied Mathematics, Chinese Academy of Sciences

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Zhongyuan Liu, Shuangjie Peng, Shusen Yan and Juncheng Wei

Problems

The incompressible Euler equations

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1}$$

are equations describing the motion of an ideal incompressible fluid(with constant density of mass).

v: the velocity, P: the pressure,
Define the corresponding vorticity $\omega = \nabla \times \mathbf{v}$, the *curl* of **v**, which describes the rotation of the fluid.

Steady vortex ring in $\mathbb{R}^3$ (from L.E.Franenkel and M.S.Berger, "A global theory of steady vortex rings in an ideal fluid", Acta Math.,132(1974), 13-51)

By the steady vortex ring we mean a figure of revolution $\mathfrak{J}$ that is expected to be homoeomorphic to a solid torus in most cases, and is associated with a continuous, axi-symmetric, solenoidal vector field v(divergence free, *div* v=0) having the following properties when we take axes fixed in the ring $\mathfrak{J}$:

1. Both $\mathfrak{J}$ and v do not vary with time;

2. the vorticity ω has positive magnitude in $\mathfrak{J}$, vanishes in $\mathbb{R}^3 \setminus \mathfrak{J}$, and satisfies a nonlinear equation of motion which, among other thing, determines the boundary of $\mathfrak{J}$;

3. v tends to a constant value at infinity in $\mathbb{R}^3$.

Example: smoke ring, "mushrooms" created by big explosions

The existence of solutions representing steady vortex rings occupies a central place in the theory of vortex motion.

For 3-dimension case, steady vortex ring in an ideal fluid can be described in the cylindrical coordinates as

$$\begin{cases} -L\Psi = 0, & \textbf{in } \{(r,z)\} \setminus A, \\ -L\Psi = \lambda r^2 f(\Psi), & \textbf{in } A, \\ \Psi(0,z) = -k \le 0, \quad \Psi|_{\partial A} = 0, \\ \Psi_r/r \to -W, \quad \Psi_z/r \to 0, \quad r^2+z^2 \to \infty, \end{cases} \tag{2}$$

where $L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$**.**

Helmholtz[1858, J.Reine Angew.Math]: first mentioned the problem;

Hill[1894,Philos.Trans.Roy.Soc.London]: discovered an Hill's type "vortex ring": a ball in $\mathbb{R}^3$;

Berger-Fraenkel [1974, Acta Math.; 1980, CMP]: Global existence of vortex rings was first established with λ as a Lagrange parameter, f convex and smooth;

Ni [1980, J.Anal.Math.], Using Mountain Pass lemma, more general f

Ambrosetti-Mancini [1981,Nonli.Anal.], **Struwe** [1988, Acta. Math.], **Ambrosetti-Struwe** [1989,ARMA], **Ambrosetti-Yang** [1990, MMMAS]: f super-linear.

Turkington [1983, CPDE] obtained a solution by studying an integral equation and analyzed the asymptotic behavior for a sequence of $\lambda \to +\infty$.

We will focus on

1. Approximating planar point vortex:
Find a family of smooth steady solutions such that the vortex set (the set where ω is not 0) shrinks to one or a couple of points.

2. Existence of planar vortex patch:
Find a family of steady solutions such that the vorticity ω for each solution is a constant λ in a connected domain Ω_λ which shrinks to a single point or several points as $\lambda \to \infty$, while $\omega = 0$ elsewhere.

We will see that they are related to the following problems:

$$\begin{cases} -\Delta\psi = \lambda(\psi)_+^p, & x \in \Omega, \\ \psi = \psi_0, & x \in \partial\Omega, \end{cases}$$

where $p > 1$**,**
and

$$\begin{cases} -\Delta\psi = \lambda 1_{\{\psi>0\}}, & x \in \Omega, \\ \psi = \psi_0, & x \in \partial\Omega. \end{cases}$$

Remark: The second one can be taken as $p = 0$ **in the first one.**

In $\mathbb{R}^2$, the vorticity $\omega := \nabla \times \mathbf{v} = \partial_1 v_2 - \partial_2 v_1$,

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0.$$

Suppose that ω is known then by Biot-Savart law

$$\mathbf{v} = -\omega * \frac{1}{2\pi} \frac{x^\perp}{|x|^2},$$

where $x^\perp = (x_2, -x_1)$ if $x = (x_1, x_2)$.

One special singular solution is $\omega = \sum_{i=1}^{m} \kappa_i \delta_{x_i(t)}$(where κ_i is the vorticity strength), which is related

$$\mathbf{v} = -\sum_{i=1}^{m} \frac{\kappa_i}{2\pi} \frac{(x - x_i(t))^\perp}{|x - x_i(t)|^2},$$

$x_i : \mathbb{R} \to \mathbb{R}^2$ **satisfy the following Kirchhoff law:**

$$\kappa_i \frac{dx_i}{dt} = (\nabla_{x_i} \mathcal{W})^\perp$$

where the Kirchhoff-Routh (path) function

$$\mathcal{W}(x_1, \cdots, x_m) := \sum_{i \neq j}^{m} \frac{\kappa_i \kappa_j}{2\pi} \log \frac{1}{|x_i - x_j|}.$$

For bounded domain $\Omega \subset \mathbb{R}^2$, similar singular solutions also exist. Suppose that the normal component of v vanishes on $\partial\Omega$, then the Kirchhoff-Routh function is

$$\mathcal{W}(x_1, \cdots, x_m) = \sum_{i \neq j}^{m} \kappa_i \kappa_j G(x_i, x_j) - \sum_{i=1}^{m} \kappa_i^2 H(x_i), \tag{3}$$

where G is the Green function of $-\Delta$ on Ω with 0 Dirichlet boundary condition and h is its regular part ($H(x) = h(x, x)$ is the Robin function),

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y|} - h(x, y).$$

Let v_n be the outward component of the velocity v on the boundary $\partial\Omega$, then we see from divergence theorem $\int_\Omega \nabla \cdot \mathbf{v} = \int_{\partial\Omega} v_n dS$ that $\int_{\partial\Omega} v_n = 0$ since $\nabla \cdot \mathbf{v} = 0$.
Let ψ_0 be determined up to a constant by

$$\begin{cases} -\Delta\psi_0 = 0, \ \textbf{in}\, \Omega, \\ -\dfrac{\partial\psi_0}{\partial\tau} = v_n, \ \textbf{on}\, \partial\Omega, \end{cases} \tag{4}$$

where $\frac{\partial\psi_0}{\partial\tau}$ denotes the tangential derivative on $\partial\Omega$.

The Kirchhoff-Routh function associated to the vortex dynamics becomes(C.C.Lin in 1941)

$$\mathcal{W}_m(x_1, \cdots, x_m) = \sum_{i \neq j}^{m} \kappa_i \kappa_j G(x_i, x_j) - \sum_{i=1}^{m} \kappa_i^2 H(x_i) + 2 \sum_{i=1}^{m} \kappa_i \psi_0(x_i). \tag{5}$$

The Kirchhoff-Routh function $\mathcal{W}_m$ **in** (5) **induces** m_1 **anti-clockwise vortices motion if all** $\kappa_i > 0$ **for** $i = 1, \cdots, m_1$, **and** $m - m_1$ **clockwise vortices motion if** $\kappa_j < 0$ **for** $j = m_1 + 1, \cdots, m$**.**

It is known that critical points of the Kirchhoff-Routh function $\mathcal{W}$ give rise to stationary vortex points solutions of the Euler equations.

Existence of critical points of $\mathcal{W}_m$ given by (5) has been studied by

T.Bartsch, A.Pistoia and T.Weth, M.del Pino, M.Kowalczyk and M.Musso, D.Bartolucci and A.Pistoia, $\cdots$

Question: For a given critical point of $\mathcal{W}$, can we get stationary smooth solutions approximating solutions concentrating near that point?

The vorticity method consists in finding maximizer of the kinetic energy defined by

$$\frac{1}{2}\int_\Omega\int_\Omega \omega(x)G(x,y)\omega(y)dxdy+\int_\Omega \psi_0(x)\omega(x)dx+\frac{1}{2}\int_\Omega |\nabla\psi_0(x)|^2dx$$

under some constraints on the sublevel sets of ω defined by

$$K_\lambda(\Omega)=\left\{\omega\in L^\infty(\Omega)\ :\ \int_\Omega \omega(x)dx=1, 0\le\omega(x)\le\lambda\ \ a.e.\ x\in\Omega\right\}.$$

In this way, B.Turkington studied the case that the flow is everywhere tangential to the boundary, i.e., $v_n = 0$**, equivalently,** $\psi_0 \equiv 0$**. He obtained an absolute maximizer** ω_λ **of**

$$max\left\{\int_\Omega\int_\Omega \omega(x)G(x,y)\omega(y)dxdy\ :\ \omega\in K_\lambda(\Omega)\right\}.$$

Indeed, $\omega_\lambda = \lambda 1_D, D = \{x\in\Omega\ :\ \psi_\lambda(x) > 0\}$**, where** $\lambda > 0$**,** ψ_λ **is the corresponding stream function satisfying**

$$\begin{cases}-\Delta\psi = \lambda 1_{\{\psi(x)>0\}}, & \text{in }\Omega,\\ \psi = \mu_\lambda, & \text{on }\partial\Omega,\end{cases} \tag{6}$$

where μ_λ **is a constant depending on** λ **such that** $\mu_\lambda = -\log\lambda + 0(1)$ **for** λ **large.**

It was showed that as $\lambda \to \infty$

$$\begin{cases} \omega_\lambda(x) \to \delta(x - x^*), & \text{in the sense of distribution} \\ \\ \psi_\lambda(x_\lambda + \frac{1}{\sqrt{\pi\lambda}} y) \to W(y), & \text{in } C^1_{loc}(\mathbb{R}^2), \end{cases} \tag{7}$$

where $x_\lambda \to x^* \in \Omega$ and $H(x^*) = max\{H(x) \, : \, x \in \Omega\}$,

$$W(y) = \begin{cases} \frac{1}{4}(1 - |y|^2), & 0 \le |y| \le 1, \\ \frac{1}{2} \log |y|^{-1}, & |y| \ge 1. \end{cases} \tag{8}$$

The stream function method

In the planar case, since $\nabla \cdot \mathbf{v} = 0$(divergence free), we can find a function(stream function) ψ such that $\mathbf{v} = (\nabla\psi)^{\perp}$, where $(\nabla\psi)^{\perp} := (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1})$. Assume that $\omega = \lambda f(\psi)$ for some function $f \in C^1(\mathbb{R})$ and λ, then

$$\begin{cases} -\Delta\psi = \lambda f(\psi), & x \in \Omega, \\ \psi = \psi_0, & x \in \partial\Omega, \end{cases}$$

f is called vorticity function, λ vortex parameter.
If ψ satisfies the above equation, then $\mathbf{v} = (\nabla\psi)^{\perp}$ and $P = F(\psi) - \frac{1}{2}|\nabla\psi|^2$ is a stationary solution to the Euler equations, where $F(t) = \int_0^t f(s)ds$.

$\mathbf{v}$ is irrotational on the set $\{x : f(\psi) = 0\}$.

$\{x : f(\psi) > 0\}$ is called the vorticity set.

Let $f(t) = 0,\ t \leq 0$. Set $q = -\psi_0$ and $u = \psi - \psi_0$, then u satisfies the following boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u - q), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

The boundary of the set $\{x : f(u - q) > 0\}$ is not known a priori and thus is a free boundary.

(9) **has been studied extensively for Ω bounded or Ω is the whole space.**

We will focus on the planar vortex patch problem:
Find a flow such that the vorticity ω is a constant λ in a connected domain Ω_λ which shrinks to a single point or several points as $\lambda \to \infty$, while $\omega = 0$ elsewhere.

To do this we need to consider the case

$$\begin{cases} -\Delta u = \lambda 1_{\{u(x) > \frac{\kappa \ln \lambda}{4\pi}\}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{10}$$

We will see that

$$\Omega_\lambda = \{x \in \Omega,\ u(x) > \frac{\kappa \ln \lambda}{4\pi}\}.$$

or

$$\begin{cases} -\Delta u = \lambda \sum_{j=1}^{k} 1_{B_\delta(x_{0,j})} 1_{\left\{u > \frac{\kappa_j \ln \lambda}{4\pi}\right\}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $\kappa_j \geq \kappa > 0$ **is some given constant, and** $\delta > 0$ **is chosen small so that** $B_\delta(x_{0,i}) \cap B_\delta(x_{0,j}) = \emptyset$**,** $i \neq j$**.**

G. Li, S. Yan and J. Yang, An elliptic problem related to planar vortex pairs, *SIAM J. Math. Anal.*, 36(2005), 1444–1460.

Their main objective is to obtain the existence result and investigate the asymptotic behavior of the solution pair (u_λ, A_λ) **of problem** (9) **as** $\lambda \to \infty$**, where**

$$A_\lambda = \{x \in \Omega : u_\lambda(x) > q(x)\}.$$

Under some standard conditions for f, using the mountain pass theorem, they obtained the least energy solution u_λ. In addition, let $q_0(x) = q(x)$ for $x \in \partial\Omega$, and suppose that $q_0(x)$ is not a constant, then for λ large, A_λ is connected and

$$diam(A_\lambda) \to 0.$$

Suppose that A_λ shrinks to a point $x_0 \in \bar{\Omega}$, then $x_0 \in \partial\Omega$ and $q(x_0) = min_{x\in\partial\Omega} q_0(x)$.

On the other hand, let x_0 be a given strict local minimum point of q on the boundary, then they obtained existence of a solution u_λ such that the corresponding set A_λ is connected and $diam(A_\lambda) \to 0$. Moreover $diam(A_\lambda)$ shrinks to x_0 as $\lambda \to \infty$.

Remark.

$$\int_\Omega \omega_\lambda = \int_\Omega \lambda f(u_\lambda - q) \to 0, \quad as\ \lambda \to \infty.$$

More interesting case is that

$$\int_\Omega \omega_\lambda = \int_\Omega \lambda f(u_\lambda - q) \to \kappa \neq 0, \quad as\ \lambda \to \infty.$$

D. Smets and J. Van Schaftingen [Desingulariation of vortices for the Euler equation, *Arch. Rat. Mech. Anal.*, 198(2010), 869–925.] investigated the following problem

$$\begin{cases} -\Delta u = \varepsilon^{-2}\left(u - q - \frac{\kappa}{2\pi}\log\frac{1}{\varepsilon}\right)_+^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{12}$$

where $p > 1$**.**

Define

$$E_\varepsilon(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{(p+1)\varepsilon^2}(u - q - \frac{\kappa}{2\pi}\log\frac{1}{\varepsilon})_+^{p+1} \right),$$

$$\mathcal{N}_\varepsilon = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \,:\, \langle dE_\varepsilon(u),\, u\rangle = 0 \right\},$$

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} E_\varepsilon(u).$$

They showed that assume that $q + \frac{\kappa}{2\pi} log \frac{1}{\varepsilon} \geq 0$ **on** Ω**, then** $\mathcal{N}_\varepsilon \neq \emptyset$ **and there exists** $u_\varepsilon \in \mathcal{N}_\varepsilon$ **such that** $E_\varepsilon(u_\varepsilon) = c_\varepsilon$**,** u_ε **is a positive solution of** (12)**.**

For a solution u_ε **of** (12)**, set**

$$\mathbf{v}_\varepsilon = (\nabla(u_\varepsilon - q))^\perp,$$

$$P_\varepsilon = \frac{1}{p+1}\left(u_\varepsilon - q - \frac{\kappa|\log\varepsilon|}{2\pi}\right)_+^{p+1} - \frac{1}{2}|\nabla(u_\varepsilon - q)|^2.$$

Then $(\mathbf{v}_\varepsilon, P_\varepsilon)$ **forms a stationary solution for problem** (1)**.**

The corresponding vorticity is $\omega_\varepsilon = \nabla \times \mathbf{v}_\varepsilon$**.**

Concerning steady solution for Euler equation they obtained:
Theorem A (Smets and Schaftingen). **Suppose that** $\Omega \subset \mathbb{R}^2$ **is a bounded simply-connected smooth domain. Let** $v_n : \partial\Omega \to \mathbb{R}$ **be such that** $v_n \in L^s(\partial\Omega)$ **for some** $s > 1$ **satisfying** $\int_{\partial\Omega} v_n = 0$**. Let** $\kappa_1 > 0$ **be given. Then, there exists** $\varepsilon_0 > 0$**, such that for each** $\varepsilon \in (0, \varepsilon_0)$**, problem** (1) **has a stationary solution** $\mathbf{v}_\varepsilon$ **with outward boundary flux given by** v_n**, and its vorticities** ω_ε **satisfies that** $supp(\omega_\varepsilon) \subset B(x_\varepsilon, C\varepsilon)$ **for some** $x_\varepsilon \in \Omega$ **and** $C > 0$ **independent of** ε **such that as** $\varepsilon \to 0$

$$\int_\Omega \omega_\varepsilon \to \kappa_1,$$

$$\mathcal{W}_1(x_\varepsilon) \to \sup_{x \in \Omega} \mathcal{W}_1(x).$$

Concerning regularization of pairs of vortices, they studied the following problem

$$\begin{cases} -\Delta u = \varepsilon^{-2}\left(u - q - \frac{\kappa_1}{2\pi}\log\frac{1}{\varepsilon}\right)_+^p - \varepsilon^{-2}(q + \frac{\kappa_2}{2\pi}\log\frac{1}{\varepsilon} - u)_+^p, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{13}$$

where $\kappa_1 > 0$, $\kappa_2 < 0$, $\varepsilon > 0$**.**

Define

$$E_\varepsilon^\pm(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - \frac{1}{(p+1)\varepsilon^2}(u - q - \frac{\kappa_1}{2\pi}\log\frac{1}{\varepsilon})_+^{p+1} - \frac{1}{(p+1)\varepsilon^2}(q + \frac{\kappa_2}{2\pi}\log\frac{1}{\varepsilon} - u)_+^{p+1}),$$

$$\mathcal{M}_\varepsilon = \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \ : u_+ \not\equiv 0,\ u_- \not\equiv 0,$$

$$\langle dE_\varepsilon^\pm(u),\, u_+\rangle = \langle dE_\varepsilon^\pm(u),\, u_-\rangle = 0\,\Big\},$$

$$d_\varepsilon = \inf_{u \in \mathcal{M}_\varepsilon} E_\varepsilon^\pm(u).$$

They obtained that:

Theorem B (Smets and Schaftingen). Suppose that Ω and v_n satisfy the same assumptions as in Theorem A. Let $\kappa_1 > 0, \kappa_2 < 0$ be given. Then, there exists $\varepsilon_0 > 0$, such that for each $0 < \varepsilon < \varepsilon_0$, problem (1) has a stationary solution $\mathbf{v}_\varepsilon$ with outward boundary flux given by v_n, and its vorticities ω_ε satisfying for some $x_\varepsilon^+, x_\varepsilon^- \in \Omega$ and $C > 0$ independent of ε,

$$supp(\omega_\varepsilon^+) \subset B(x_\varepsilon^+, C\varepsilon), \; supp(\omega_\varepsilon^-) \subset B(x_\varepsilon^-, C\varepsilon).$$

Moreover as $\varepsilon \to 0$

$$\int_\Omega \omega_\varepsilon^+ \to \kappa_1, \; \int_\Omega \omega_\varepsilon^- \to \kappa_2,$$

$$\mathcal{W}_2(x_\varepsilon^+, x_\varepsilon^-) \to \sup_{x^+, x^- \in \Omega} \mathcal{W}_2(x^+, x^-).$$

"Regularization of point vortices pairs for the Euler equation in dimension two" by *D.Cao, Z.Liu and J.Wei (ARMA,2014)*

Theorem 1. Under the same assumptions as in Theorem A. For any given $\kappa_i > 0 (i = 1, \cdots, m)$ and a C^1 stable critical point $(x_{1,*}, \cdots, x_{m,*})$ of Kirchhoff - Routh function $\mathcal{W}_m(x_1, \cdots, x_m)$ defined by (5), there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (1) has a stationary solution $\mathbf{v}_\varepsilon$ with outward boundary flux given by v_n, whose vorticity can be represented by $\omega_\varepsilon = \sum_{i=1}^{m} \omega_{i,\varepsilon}$ satisfying for small ε,

$$supp(\omega_{i,\varepsilon}) \subset B(x_{i,\varepsilon}, C\varepsilon), \qquad \textbf{for } i = 1, \cdots, m,$$

where $x_{i,\varepsilon} \in \Omega_i$ $(i = 1, \cdots, m)$**,** $C > 0$ **is a constant independent of** ε**. Furthermore as** $\varepsilon \to 0$**,**

$$\int_{B(x_{i,\varepsilon}, C\varepsilon)} \omega_{i,\varepsilon} \to \kappa_i, \qquad i = 1, \cdots, m,$$

$$\int_\Omega \omega_\varepsilon \to \sum_{i=1}^m \kappa_i,$$

$$(x_{1,\varepsilon}, \cdots, x_{m,\varepsilon}) \to (x_{1,*}, \cdots, x_{m,*}).$$

Theorem 2. **Under the same assumptions as in Theorem A, for any given $\kappa_i > 0(i = 1, \cdots, m_1)$, $\kappa_i < 0(i = m_1 + 1, \cdots, m)$ and a C^1 stable critical point $(x_{1,*}, \cdots, x_{m,*})$ of Kirchhoff - Routh function $\mathcal{W}_m$ defined by (5), there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (1) has a stationary solution $\mathbf{v}_\varepsilon$ with outward boundary flux given by v_n whose vorticity can be represented by $\omega_\varepsilon = \sum_{i=1}^{m} \omega_{i,\varepsilon}$ satisfies for small ε,**

$$supp(\omega_{i,\varepsilon}) \subset B(x_{i,\varepsilon}, C\varepsilon), \qquad \textbf{for } i = 1, \cdots, m,$$

where $x_{i,\varepsilon} \in \Omega_i \ (i = 1, \cdots, m)$**,and** $C > 0$ **is a constant independent of** ε**. Furthermore as** $\varepsilon \to 0$**,**

$$(x_{1,\varepsilon}, \cdots, x_{m,\varepsilon}) \to (x_{1,*}, \cdots, x_{m,*}),$$

$$\int_{B(x_{i,\varepsilon}, C\varepsilon)} \omega_{i,\varepsilon} \to \kappa_i, \quad i = 1, \cdots, m,$$

$$\int_\Omega \omega_\varepsilon \to \sum_{i=1}^m \kappa_i.$$

We need to consider the following problem

$$\begin{cases} -\varepsilon^2 \Delta u = \sum_{i=1}^{m} 1_{B_\delta(x_{i,*})}\Big(u - q - \frac{\kappa_i}{2\pi} \log \frac{1}{\varepsilon}\Big)_+^p & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{14}$$

where $p > 1$ **and** $\delta > 0$ **is small so that** $B_\delta(x_{i,*})(i = 1, \cdots, m)$ **are mutually disjoint ball.**

Proof of Theorem 1

We will use a reduction argument to prove Theorem 1. To this end, we need to construct an approximate solution for (14)**. We will construct solutions for** (14) **of the form**

$$u_\varepsilon(x) = \sum_{i=1}^{m} M_{\varepsilon, z_i, a_{\varepsilon, i}}(x) + r_\varepsilon(x),$$

where $z_i \in \Omega_i$**,** $a_{\varepsilon, i} > 0$ **for** $i = 1, \cdots, m$**,**

$\sum_{i=1}^{m} M_{\varepsilon, z_i, a_{\varepsilon, i}}$ **is the main part and** r_ε **is a small perturbation term.**

We only state the result for $q \equiv 0$.

Theorem 3. **Let $v_n = 0$, $\kappa_j > 0 (j = 1, \cdots, m)$ be given. Suppose that $\mathbf{x}_0 = (x_{0,1}, \cdots, x_{0,m}) \in \Omega^m$ is an isolated critical point of $\mathcal{W}_m(\mathbf{x})$ defined by (3) satisfying $\deg(\nabla \mathcal{W}_m, \mathbf{x}_0) \neq 0$. Then there exists a $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, problem (1) has a solution $\mathbf{v}_\varepsilon$, the corresponding vorticity ω_ε of which satisfies $\omega_\varepsilon(x) = \varepsilon^{-2}$ for $x \in \bigcup_{j=1}^m \Omega_{j,\varepsilon} \subset \Omega$, and $\omega_\varepsilon(x) = 0$ elsewhere, where $\Omega_{j,\varepsilon}$ satisfies**

$$B_{\varepsilon\sqrt{\frac{\kappa_j}{\pi}}\left(1-L_1\varepsilon|\ln\varepsilon|^{\frac{3}{2}}\right)}(x_{j,\varepsilon}) \subset \Omega_{j,\varepsilon} \subset B_{\varepsilon\sqrt{\frac{\kappa_j}{\pi}}\left(1+L_1\varepsilon|\ln\varepsilon|^{\frac{3}{2}}\right)}(x_{j,\varepsilon})$$

for some $L_1 > 0$ and $x_{j,\varepsilon} \in \Omega$ near $x_{0,j}$.
Furthermore as $\varepsilon \to 0$,

$$x_{j,\varepsilon} \to x_{0,j}, \quad \int_{\Omega_{j,\varepsilon}} \omega_\varepsilon(x) = \kappa_j + o(1), \quad \text{for } j = 1, \cdots, m,$$

$$\int_\Omega \omega_\varepsilon(x) = \sum_{j=1}^m \kappa_j + o(1),$$

where $o(1)$ denotes various quantities that go to 0 as $\varepsilon \to +\infty$.

We need to study the following boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta \psi = \sum_{j=1}^{m} 1_{B_\delta(x_{0,j})} 1_{\{\psi > \frac{\kappa_j |\ln \varepsilon|}{2\pi}\}}, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $\kappa_j \geq \kappa > 0$ is some given constant, and $\delta > 0$ is chosen small so that $B_\delta(x_{0,i}) \cap B_\delta(x_{0,j}) = \emptyset$, $i \neq j$. Note that (15) corresponds to the case that the rotation of the flow at each x_i is anti-clockwise.

Remark: (15) corresponding to $p = 0$ in (14).

The main difference between the case $f(u) = (u - \kappa)_+^p (p > 1)$ and $f(u) = 1_{\{u>\kappa\}}(p = 0)$ is that in the case $p > 1$ the nonlinear function is smooth, while when $p = 0$ it is no longer continuous. The functional corresponding to (15) is non-smooth(not C^1).

When $p = 0$ the linearized operator at its approximate solution involves Dirac measures while for $p > 1$ the linearized operator at its approximate solution is an elliptic operator with smooth coefficient.

The involvement of the measure implies that we need to use domain-variation type estimates. The estimates for $p > 1$ are all carried out in some standard Sobolev spaces.

It is possible to use the critical point theory for non-smooth functional to obtain solutions. However it is hard to obtain the asymptotic behavior of solutions as $\varepsilon \to 0$.

K. C. Chang, The obstacle problem and partial differential equations with discontinuous nonlinearities, *Comm. Pure Appl. Math*. 33(1980), 117–146.

K. C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl*. 80(1981), 102–129.

We only need to study the following boundary value problem

$$\begin{cases} -\varepsilon_1^2 \Delta u = \sum_{j=1}^{m} 1_{B_\delta(x_{0,j})} 1_{\{u>\kappa_j\}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon_1 = \frac{\varepsilon \sqrt{|\ln \varepsilon|}}{2\pi}$.

Indeed $\psi = \frac{|\ln \varepsilon|}{2\pi} u$ satisfies (15).
In the sequel we will consider the following problem instead

$$\begin{cases} -\varepsilon^2 \Delta u = \sum_{j=1}^{m} 1_{B_\delta(x_{0,j})} 1_{\{u>\kappa_j\}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Let $\mathbf{x} = (x_1, \cdots, x_m) \in \Omega^m$ be a point close to $\mathbf{x}_0$, which is a non-degenerate critical point of the Kirchhoff-Routh function $\mathcal{W}_m$ defined in (3). We are try to find solutions of the form

$$\mathcal{U}_{\varepsilon,\mathbf{x},a} + r_\varepsilon, \tag{17}$$

where

$$\mathcal{U}_{\varepsilon,\mathbf{x},a} = \sum_{j=1}^{m} PU_{\varepsilon,x_j,a_j}. \tag{18}$$

and a_j is chosen suitably close to κ_j, r_ε is a perturbation term.

Question 1. **What are PU_{ε,x_j,a_j}?**

Question 2. **How to obtain r_ε?**

Answer to Question 1

Let $R > 0$ **be a large constant, such that for any** $x \in \Omega$, $\Omega \subset B_R(x)$. **Consider the following problem:**

$$\begin{cases} -\varepsilon^2 \Delta u = 1_{u>a}, & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0), \end{cases} \tag{19}$$

where $a > 0$ **is a constant. Then,** (19) **has a solution** $U_{\varepsilon,a}$

$$U_{\varepsilon,a}(y) = \begin{cases} a + \frac{1}{4\varepsilon^2}\left(s_\varepsilon^2 - |y|^2\right), & |y| \le s_\varepsilon, \\ a \ln \frac{|y|}{R} / \ln \frac{s_\varepsilon}{R}, & s_\varepsilon \le |y| \le R, \end{cases} \tag{20}$$

where s_ε **is the constant, such that** $U_{\varepsilon,a} \in C^1(B_R(0))$. **So,** s_ε **satisfies**

$$-\frac{s_\varepsilon}{2\varepsilon^2} = \frac{a}{s_\varepsilon \ln \frac{s_\varepsilon}{R}}. \tag{21}$$

For any $x \in \Omega$, define $U_{\varepsilon,x,a}(y) = U_{\varepsilon,a}(y - x)$. Because $U_{\varepsilon,x,a}$ does not vanish on $\partial\Omega$, we need to make a projection. Let $PU_{\varepsilon,x,a}$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta w = 1_{\{U_{\varepsilon,x,a} > a\}}, & \textbf{in } \ \Omega, \\ w = 0, & \textbf{on } \partial\Omega. \end{cases}$$

Then

$$PU_{\varepsilon,x,a}(y) = U_{\varepsilon,x,a}(y) - \frac{a}{\ln \frac{R}{s_\varepsilon}} g(y, x), \tag{22}$$

To get a better approximation, we need to choose $a_{\varepsilon,j}$ properly. $a_{\varepsilon,j}$ and $s_{\varepsilon,j}$, $j = 1, \cdots, m$ should satisfy the following system:

$$a_i = \kappa_i + \frac{a_i}{\ln \frac{R}{s_{\varepsilon,i}}} g(x_i, x_i) - \sum_{j \neq i} \frac{a_j}{\ln \frac{R}{s_{\varepsilon,j}}} \bar{G}(x_i, x_j), \tag{23}$$

and

$$\frac{s_{\varepsilon,i} \sqrt{\ln \frac{R}{s_{\varepsilon,i}}}}{\varepsilon} = \sqrt{2a_i}, \tag{24}$$

where $\bar{G}(y, x) = \ln \frac{R}{|y-x|} - g(y, x)$.

Our aim is to find a solution $u = \mathcal{U}_{\varepsilon,\mathbf{x},a} + r$ **for (16). So,** r **satisfies**

$$-\varepsilon^2 \Delta r = \sum_{j=1}^{m} 1_{B_\delta(x_{0,j})} 1_{\mathcal{U}_{\varepsilon,\mathbf{x},a}+r>\kappa_j} - \sum_{j=1}^{m} 1_{U_{\varepsilon,x_j,a_{\varepsilon,j}}>a_{\varepsilon,j}} \quad \text{in } \Omega. \qquad (25)$$

Let

$$E_{\varepsilon,\mathbf{x},p} = \left\{ u \in W_0^{1,p}(\Omega), \int_\Omega \nabla \frac{\partial PU_{\varepsilon,x_j,a_{\varepsilon,j}}}{\partial x_{jh}} \nabla u = 0,\ j = 1,\cdots,m,\ h = 1,2 \right\},$$

$$F_{\varepsilon,\mathbf{x},p} = \left\{ u \in W^{-1,p}(\Omega),\ \int_\Omega \frac{\partial PU_{\varepsilon,x_j,a_{\varepsilon,j}}}{\partial x_{jh}} u = 0,\ j = 1,\cdots,m,\ h = 1,2 \right\},$$

where $p > 2$ **is a fixed constant.**

Now we define the linear operator $\mathbb{L}_\varepsilon$ as follows.

$$\mathbb{L}_\varepsilon u = -\Delta u - 2\sum_{j=1}^{m} \frac{1}{s_{\varepsilon,j}} u(s_{\varepsilon,j},\theta)\delta_{|y-x_{\varepsilon,j}|=s_{\varepsilon,j}}, \quad u \in E_{\varepsilon,\mathbf{x},p}. \tag{26}$$

For any $u \in W^{-1,p}(\Omega)$, we define Q_ε as follows:

$$Q_\varepsilon u = u + \sum_{j=1}^{m}\sum_{h=1}^{2} b_{jh}\Delta \frac{\partial PU_{\varepsilon,x_j,a_j}}{\partial x_{jh}},$$

where b_{j1} and b_{j2} are the constants such that $Q_\varepsilon u \in F_{\varepsilon,\mathbf{x},p}$.

Proposition 4. $Q_\varepsilon \mathbb{L}_\varepsilon$ **is one to one and onto from** $E_{\varepsilon,\mathbf{x},p}$ **to** $F_{\varepsilon,\mathbf{x},p}$**.**

Consider

$$Q_\varepsilon \mathbb{L}_\varepsilon r = Q_\varepsilon R_\varepsilon(r), \tag{27}$$

where

$$\begin{aligned} R_\varepsilon(r) =& \frac{1}{\varepsilon^2}\Big(\sum_{j=1}^m 1_{B_\delta(x_{0,j})} 1_{\mathcal{U}_{\varepsilon,\mathbf{x},a}+r>\kappa_j} - \sum_{j=1}^m 1_{U_{\varepsilon,x_j,a_{\varepsilon,j}}>a_{\varepsilon,j}}\Big) \\ & -2\sum_{j=1}^m \frac{1}{s_{\varepsilon,j}} a(s_{\varepsilon,j},\theta)\delta_{|y-x_{\varepsilon,j}|=s_{\varepsilon,j}} r. \end{aligned} \tag{28}$$

Using Proposition 4, we can rewrite (27) **as**

$$r = G_\varepsilon r =: (Q_\varepsilon \mathbb{L}_\varepsilon)^{-1} Q_\varepsilon R_\varepsilon(r). \qquad (29)$$

Proposition 5. **Fixe a constant** $p > 2$**. There is an** $\varepsilon_0 > 0$**, such that for any** $\varepsilon \in (0, \varepsilon_0]$**,** (27) **has a unique solution** $r_{\varepsilon,\mathbf{x}} \in E_{\varepsilon,\mathbf{x},+\infty}$**, with**

$$s_{\varepsilon,j}^{1-\frac{2}{p}} \|\nabla r_{\varepsilon,\mathbf{x}}\|_{L^p(\cup_{j=1}^k B_{2Ls_{\varepsilon,j}}(x_j))} + \|r_{\varepsilon,\mathbf{x}}\|_{L^\infty(\Omega)} = O\Big(\sum_{j=1}^m \frac{s_{\varepsilon,j}}{|\ln s_{\varepsilon,j}|}\Big),$$

and

$$s_{\varepsilon,1}\|\nabla r_{\varepsilon,\mathbf{x}}\|_{L^\infty(\cup_{j=1}^m B_{2Ls_{\varepsilon,j}}(x_j))} \le \sqrt{\varepsilon}.$$

Moreover, $r_{\varepsilon,\mathbf{x}}$ **is a continuous map from x to** $E_{\varepsilon,\mathbf{x},p}$ **in the norm of** $H^1(\Omega)$**.**

Let

$$M = E_{\varepsilon,\mathbf{x},+\infty} \cap \Big\{ s_{\varepsilon,1}^{1-\frac{2}{p}} \|\nabla r\|_{L^p(\cup_{j=1}^m B_{2Ls_{\varepsilon,j}}(x_j))} + \|r\|_{L^\infty(\Omega)} \le \varepsilon,$$
$$s_{\varepsilon,1}\|\nabla r\|_{L^\infty(\cup_{j=1}^m B_{2Ls_{\varepsilon,j}}(x_j))} \le \sqrt{\varepsilon}\Big\}.$$

Main idea of the Proof of Proposition 5

Step 1. G_ε **is a map from** M **to** M**.**

Step 2. G_ε **is a contraction map.**

We will choose x, such that $\mathcal{U}_{\varepsilon,\mathbf{x},a} + r_{\varepsilon,\mathbf{x}}$ is a solution of (16), where $r_{\varepsilon,\mathbf{x}}$ is the map obtained in Proposition 5.

How to choose x?

Lemma 6. If x satisfies

$$\int_\Omega \Big(\varepsilon^2 \nabla(\mathcal{U}_{\varepsilon,\mathbf{x},a} + r_{\varepsilon,\mathbf{x}}) \nabla \frac{\partial PU_{\varepsilon,x_j,a_{\varepsilon,j}}}{\partial x_{jh}} - \sum_{i=1}^m 1_{B_\delta(x_{0,i})} 1_{\{\mathcal{U}_{\varepsilon,\mathbf{x},a}+r_{\varepsilon,\mathbf{x}}>\kappa_i\}} \frac{\partial PU_{\varepsilon,x_j,a_{\varepsilon,j}}}{\partial x_{jh}} \Big) = 0, \tag{30}$$

for $j = 1, \cdots, m$, $h = 1, 2$, then $\mathcal{U}_{\varepsilon,\mathbf{x},a} + r_{\varepsilon,\mathbf{x}}$ is a solution of (16).

(30) is equivalent to

$$\nabla \mathcal{W}_m(\mathbf{x}) = o(1). \tag{31}$$

Thank You

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