Workshop on Partial Differential Equation and its Applications

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Institute for Mathematical Sciences - National University of Singapore

The p-Laplacian and geometric structure of the Riemannian manifold

Nguyen Thac Dung (VIASM)
Vietnam Institute for Advanced Study in Mathematics

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Outline

Let (M, g) be a Riemannian manifold with metric tensor $g = g_{ij} dx_i \otimes dx_j$. The Laplacian on M is defined by

$$\Delta := -\frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x_i} \left((\det g_{ij}) g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $(g^{ij})^t = (g_{ij})^{-1}$. It is well-known that

1. Δ is a self-adjoint operator.
2. $0 < \lambda_1 \leq \lambda_2 < \ldots < \lambda_k < \ldots, \lim\limits_{k \to \infty} \lambda_k = \infty$, where λ_k is the k-th eigenvalue of Δ

$$\Delta \phi_k = -\lambda_k \phi_k$$

for some $0 \not\equiv \phi \in \mathcal{C}^\infty(M)$.

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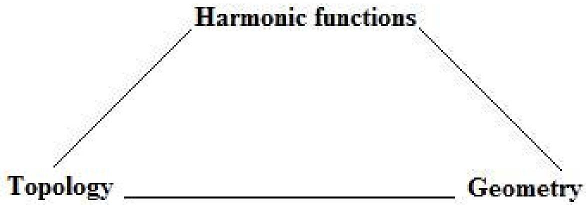
Harmonic functions
Topology
Geometry

A panoramic view of Riemannian Geometry - Berger

"Already in the middle ages bell makers knew how to detect invisible cracks by sounding a bell on the ground before lifting it up to the belfry. How can one test the resistance to vibrations of large modern structures by nondestructive essays?... A small crack will not only change the boundary shape of our domain, one side of the crack will strike the other during vibrations invalidating our use of the simple linear wave equation"

Schuster 1882

Can we hear the shape of a drum?

Answer: No by Gordon, Web and Wolpert (1982).

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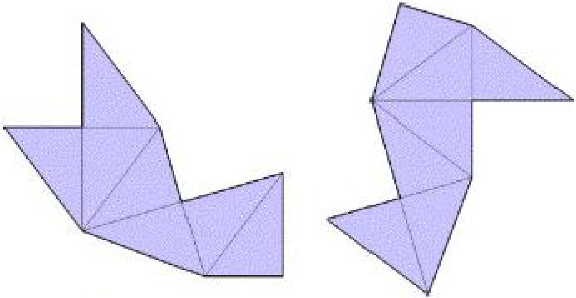

Figure: Two domains with the same eigenvalues

Let Ω be a polygon in $\mathbb{R}^n$

Mark Kac - 1966

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{1}{4\pi t}\left(area(\Omega) - \sqrt{4\pi t} length(\Omega) + \frac{2\pi t}{3}(1-\gamma(\Omega))\right),$$

where $\gamma(\Omega)$ is the genus of Ω.

1967, McKean and Singer: Riemannian manifolds with boundary.
1953, Minakshisundaram: compact Riemmanian manifold M without boundary,

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{1}{(4\pi t)^{n/2}}\left(Vol(M) - \frac{t}{6}\int_M R_g dv_g + O(t^2)\right),$$

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Question:

How about "the shape" of a complete non-compact Riemannian manifold?

Let M be a complete Riemannian manifold.
$\lambda_1(M)$: the lower bound of the spectrum of the Laplacian on M

$$\lambda_1(M) = \inf\left\{\frac{\int_M |\nabla\phi|^2}{\int_M \phi^2} : \phi \in \mathcal{C}_0^\infty(M)\right\}.$$

Informations on $\lambda_1(M) \Longrightarrow$ geometry of the manifold.

1. Analytic properties implies geometric properties.
2. Geometric structures of complete Riemannian manifolds.

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Li-Wang: Jour. Diff. Geom., 2002

Let M be an m-dimensional complete Riemannian manifold with $n \geq 4$. Suppose that

$$\mathrm{Ric}_M \geq -(n-1)$$

and

$$\lambda_1(M) \geq \frac{(n-1)^2}{4}.$$

Then either:

1. M has only one end; or
2. $M = \mathbb{R} \times N$ with warped product metric

$$ds_M^2 = dt^2 + e^{2t} ds_N^2$$

where N is a compact manifold with nonnegative Ricci curvature.

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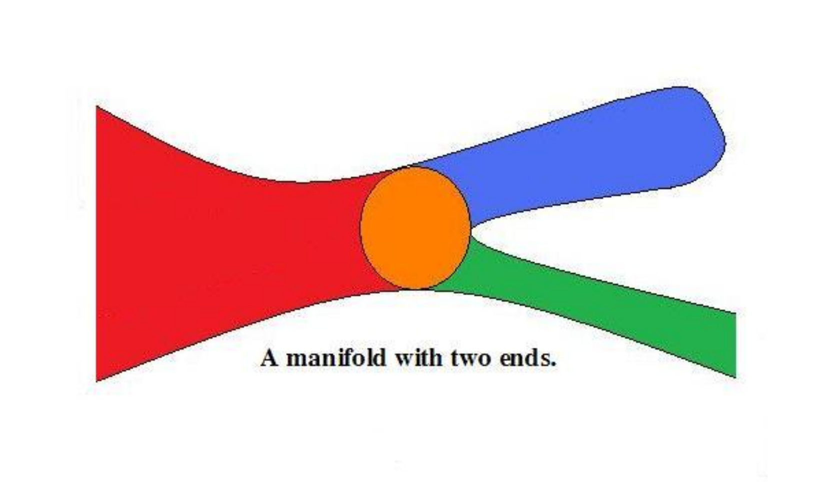
A manifold with two ends.

For $p > 1$, the p-Laplacian is defined by

$$\Delta_p f := div(|\nabla f|^{p-2}\nabla f).$$

Let $\lambda_1(\Delta_p)$ be the smallest eigenvalue of Δ_p in M.

$$\lambda_1(\Delta_p) = \inf\left\{\frac{\int_M |\nabla u|^p}{\int_M |u|^p}, u \int \mathcal{C}_0^\infty(M)\right\}.$$

1. Moser (J.Eur. M.S. 2007): Inverse mean curvature flow
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Theorem (Sung-Wang, MRL 2014)

Let (M^n, g) be a complete manifold of dimension $n \geq 2$. Suppose that

$$Ric_M \geq -(n-1)$$

and

$$\lambda_1(\Delta_p) = \left(\frac{n-1}{p}\right)^p.$$

Then either

1. M has no finite volume ends; or
2. $M = \mathbb{R} \times N$ with warped product metric

$$ds_M^2 = dt^2 + e^{2t} ds_N^2$$

and N being a compact manifold with nonnegative Ricci curvature.

M. Batista, M. P. Cavalcante, N. L. Santos (Geo. Dedi. 2014)

1. M^{2m}: complete non-compact Kähler manifold with holomorphic bisectional curvature $BK_M \geq -1$ then
$$\lambda_1(\Delta_p) \leq \frac{4^p m^p}{p^p}.$$
2. M^{4m}: complete non-compact quaternionic Kähler manifold with scaltar curvature $S_M \geq -16m(m+2)$ then
$$\lambda_1(\Delta_p) \leq \frac{2^p (2m+1)^p}{p^p}.$$

Theorem 1

Let M^{2m} be a complete Kähler manifold of complex dimension $m \geq 1$ with holomorphic bisectional curvature bounded by

$$BK_M \geq -1.$$

If $\lambda_{1,p} \geq \left(\frac{2m}{p}\right)^p$, then either

1. M has no p-parabolic end; or
2. M splits as a warped product $M = \mathbb{R} \times N$ where N is a compact manifold. Moreover, the metric is given by

$$ds_M^2 = dt^2 + e^{-4t}\omega_2^2 + e^{-2t}\sum_{\alpha=3}^{2m}\omega_\alpha^2,$$

where $\{\omega_2, \ldots, \omega_{2m}\}$ are orthonormal coframes for N.

Theorem 2

Let M^{4m} be a complete noncompact quaternionic Kähler manifold of real dimension $4m$ with the scalar curvature of M bounded by

$$S_M \geq -16m(m+2).$$

If $\lambda_{1,p} \geq \left(\frac{2(2m+1)}{p}\right)^p$, then either

1. M has no p-parabolic end; or
2. M splits as a warped product $M = \mathbb{R} \times N$ where N is a compact manifold. Moreover, the metric is given by

$$ds_M^2 = dt^2 + e^{4t} \sum_{p=2}^{4} \omega_p^2 + e^{2t} \sum_{\alpha=5}^{4m} \omega_\alpha^2,$$

where $\{\omega_2, \ldots, \omega_{4m}\}$ are orthonormal coframes for N.

Let $E \subset M$ be an end. E is p-parabolic if it does not admit a p-harmonic function $f : E \to \mathbb{R}$ such that

$$\begin{cases} f\,|_{\partial E} = 1; \\ \liminf_{y \in E, y \to \infty} f(y) < 1. \end{cases}$$

Remark

- Li-Wang (Amer. Jour. Math, 2009) proved the Theorem 1, when $p = 2$
- Kong-Li-Zhou (JDG, 2008) proved the Theorem 2, when $p = 2$.

Suppose that M has a p-parabolic end E. Let β be the Busemann function associated with a geodesic ray γ contained in E, namely,

$$\beta(q) = \lim_{t\to\infty} \left(t - dist(q, \gamma(t))\right).$$

The Laplacian comparison theorems (by LW, KLZ) imply

$$\Delta\beta \geq -a,$$

where $a = 2m$ in theorem 1 (Li-Wang) and $a = 2(2m+1)$ in theorem 2 (Kong-Li-Zhou).

Claim

$$\Delta\beta = -a$$

Idea: for $b = \frac{a}{p}$, let $g = e^{b\beta}$, we can show

$$\Delta_p(g) \geq -\lambda_{1,p} g^{p-1}.$$

Using the volume estimate of S. Buckley and P. Koskela (Matth. Zeits., 2005), it turns out that

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The p-Laplacian on smooth metric measure spaces

A smooth metric measure space $(M, g, e^{-f}dv)$ contains

- a Riemannian manifold: (M, g)
- a weighted volume form: $e^{-f}dv$, where $f \in \mathcal{C}^{\infty}(M)$.
- dv is the volume element induced by the Riemannian metric g.

For $p > 1$, a function u is said to be weighted p-harmonic if $\Delta_{f,p} u := e^f div(e^{-f}|\nabla u|^{p-2}\nabla u) = 0$.

- Note that weighted p-harmonic functions are characterized as the critical points of the weighted Dirichlet energy

$$\int_M |\nabla u|^p e^{-f} dv.$$

If u is a weighted p-harmonic function and

$$\int_M |\nabla u|^p e^{-f} dv < +\infty$$

then u is said to have finite p-energy.
The bottom spectrum of the weighted Laplacian $\Delta_{f,p}$,

$$\lambda_{1,p} := \inf \left\{ \frac{\int_M |\nabla \phi|^p e^{-f} dv}{\int_M \phi^p e^{-f} dv}, \phi \in C_0^\infty(M) \right\}.$$

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For any $m \in \mathbb{N}$, the m dimensional Bakry-Émery curvature Ric_f^m associated to smooth metric measure space $(M, g, e^{-f}dv)$ is defined by

$$Ric_f^m = Ric + Hess(f) - \frac{\nabla f \otimes \nabla f}{m - n},$$

where

- Ric denotes the Ricci curvature of (M, g), and $Hess(f)$ the Hessian of f.
- $m \geq n$. $m = n$ if and only if f is constant.

We also define

$$Ric_f := Ric + Hess(f)$$

Proposition (Y. Wang and J. Y. Chen, 2014)

Let $\widetilde{g} = e^{-\frac{2f}{n-p}}$ be a conformal change of g. Then u is a weighted p-harmonic function on (M, g) if and only if u is a p-harmonic function on $(M, \widetilde{g})$.

Theorem 3 (Local gradient estimate)

Let (M^n, g, e^{-f}) be a smooth metric measure space of dimension n with $Ric_f^m \geq -(m-1)\kappa$. Suppose that v is a positive smooth weighted p-harmonic function on the ball $B_R = B(o, R) \subset M$. Then there exists a constant $C = C(p, m, n)$ such that

$$\frac{|\nabla v|}{v} \leq \frac{C(1+\sqrt{\kappa}R)}{R} \quad \text{on} \quad B(o, R/2).$$

Ideas of proof.

1. Laplacian comparison

$$\Delta_f \rho := \Delta - \langle \nabla f, \nabla \rho \rangle \leq (m-1)\sqrt{\kappa}\coth(\sqrt{\kappa}\rho)$$

   provided that $Ric_f^m \geq -(m-1)\kappa$. Here $\rho(x) := dist(o, x)$ stands for the distance between $x \in M$ and a fixed point $o \in M$.

2. Volume comparison

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq \frac{V_{\mathbb{H}^m}(r_2)}{V_{\mathbb{H}^m}(r_1)}$$

3. Sobolev inequality.

Sobolev inequality

Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete noncompact smooth metric measure space. If $\mathrm{Ric}_f^m \geq -(m-1)\kappa$ for some nonnegative constants κ, then for any $p > 1$, there exists a constant $c = c(n, p, m) > 0$ depending only on p, n, m such that

$$\left(\int_{B_R} |\varphi|^{\frac{2p}{p-2}} e^{-f}\right)^{\frac{p-2}{p}} \leq \frac{R^2.e^{c(1+\sqrt{\kappa}R)}}{V_f(B_R)^{\frac{2}{p}}} \int_{B_R} \left(|\nabla\varphi|^2 + R^{-2}\varphi^2\right) e^{-f}$$

for any $\varphi \in C_0^\infty(B_p(R))$.

By theorem 3,

$$\frac{|\nabla v|}{v} \leq \frac{C(1+\sqrt{\kappa}R)}{R} \quad \text{on} \quad B(o, R/2).$$

Harnack inequality

Let $(M, g, e^{-f}dv)$ be a complete smms of dimension $n \geq 2$ with $Ric_f^m \geq -(m-1)\kappa$. Suppose that v is a positive weighted p-harmonic function on the geodesic ball $B(o, R) \subset M$. There exists a constant $C_{p,n,m}$ depending only on p, n, m such that

$$v(x) \leq e^{C_{p,n,m}(1+\sqrt{\kappa}R)}v(y), \qquad \forall x, y \in B(o, R/2).$$

If $\kappa = 0$, we have a uniform constant $c_{p,n,m}$ (independent of R) such that

$$\sup_{B(o,R/2)} v \leq c_{p,n} \inf_{B(o,R/2)} v.$$

Liouville property

Assume that (M, g) is a smms with $\mathrm{Ric}_f^m \geq 0$. If u is a weighted p-harmonic function bounded from below on M then u is constant.

Estimate of the first weighted eigenvalue

Let $(M^n, g, e^{-f}dv)$ be an n-dimensional complete noncompact manifold with $Ric_f^m \geq -(m-1)$. Then

$$\lambda_{1,p} \leq \left(\frac{m-1}{p}\right)^p.$$

1. Volume estimate of Buckley and Koskela,

$$V_f(B(r)) \geq Ce^{p\lambda_{1,p}^{1/p} r}, r \text{ large enough}$$

2. Volume comparison theorem

$$V_f(B(r)) \leq C_1 e^{(m-1)r}.$$

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Theorem 4 (Global sharp gradient estimate)

Let $(M^n, g, e^{-f}dv)$ be an n-dimensional complete noncompact manifold with $Ric_f^m \geq -(m-1)$. If v is a positive weighted p-eigenfunction with respect to the first eigenvalue $\lambda_{1,p}$, that is,

$$e^f div(e^{-f}|\nabla v|^{p-2}\nabla v) = -\lambda_{1,p}v^{p-1}$$

then

$$|\nabla \ln v| \leq y.$$

Here y is the unique positive root of the equation

$$(p-1)y^p - (m-1)y^{p-1} + \lambda_{1,p} = 0.$$

Corollary

Let $(M^n, g, e^{-f}dv)$ be an n-dimensional complete noncompact manifold with $Ric_f^m \geq -(m-1)$. If v is a positive weighted p-harmonic function then

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Let $M^n = \mathbb{R} \times N^{n-1}$ with a warped product metric

$$ds^2 = dt^2 + e^{2t} ds_N^2,$$

where N is a complete manifold with non-negative Ricci curvature. Then we can compute

$$Ric_M \geq -(n-1)$$

$$\Delta = \frac{\partial^2}{\partial t^2} + (n-1)\frac{\partial}{\partial t} + e^{-2t}\Delta_N.$$

Choose weighted function $f = (m-n)t$, then $Ric_f^m \geq -(m-1)$.
Let $v(t,x) = e^{-at}$, where $\frac{m-1}{p} \leq a \leq \frac{m-1}{p-1}$, we have

$$\frac{|\nabla v|}{v} = a.$$

$$e^f div(e^{-f}|\nabla v|^{p-2}\nabla v) = ((p-1)a - (m-1))a^{p-1}v^{p-1}.$$

This implies that

$\lambda_{1,p} = (m-1-(p-1)a)a^{p-1}$, or $(p-1)a^p - (m-1)a^{p-1} + \lambda_{1,p} = 0.$

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Theorem 5

Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space of dimension $n \geq 2$. Suppose that $Ric_f^m \geq -(m-1)$ and $\lambda_{1,p} = \left(\frac{m-1}{p}\right)^p$. Then either M has no p-parabolic ends or $M = \mathbb{R} \times N^{n-1}$ for some compact manifold N.

Suppose that M has a p-parabolic end E. Let β be the Busemann function associated with a geodesic ray γ containd in E, namely,

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Theorem 6 (Rigidity - D)

Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space of dimension $n \geq 3$. Suppose that $Ric_f^m \geq -(m-1)$ and $\lambda_{1,p} = \left(\frac{m-1}{p}\right)^p$ for some $2 \leq p \leq \frac{(m-1)^2}{2(m-2)}$. Then either M has only one p-nonparabolic end or $M = \mathbb{R} \times N^{n-1}$ for some compact manifold N.

An open and interesting problem

Can we get the same results if $Ric_f \geq -(m-1)$.

Theorem 6 (Rigidity - D)

Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space of dimension $n \geq 3$. Suppose that $Ric_f^m \geq -(m-1)$ and $\lambda_{1,p} = \left(\frac{m-1}{p}\right)^p$ for some $2 \leq p \leq \frac{(m-1)^2}{2(m-2)}$. Then either M has only one p-nonparabolic end or $M = \mathbb{R} \times N^{n-1}$ for some compact manifold N.

An open and interesting problem

Can we get the same results if $Ric_f \geq -(m-1)$.


thank you!